BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Tomul L (LIV), Fasc. 1–4, 2004
CONSTRUCŢII. ARHITECTURĂ

D.C. 624.07; 624.1

A COMPARATIVE STUDY OF SOLUTIONS CONCERNING THICK ELASTIC PLATES ON BI-MODULUS FOUNDATION

BY

OLIMPIA BLĂGOI and *IOANA VLAD

The classical bending theory of elastic plates is based upon the assumption that the internal moments are proportional to the curvatures of the median deformed surface. This theory does not include the effects of shear and normal pressure in the plate. The model of a bi-modulus foundation is a realistic generalization of the Winkler's classical one and is widely used to represent the subgrade of railroad systems, airport lanes [1], [2]. The derived equation of elastic thick plates on bi-modulus foundation considers shear and normal stress as linear variable across the plate thickness.

This paper presents numerical solutions for thick plate resting on bi-modulus subgrade. These solutions take into account the shear distortion, and they are compared to the solutions obtained by Finite Element Analysis and with the Winkler's model. Particular solutions for the rectangular plate of clamped boundary, for the hinged rectangular plate and for a semi-elliptical plate, are discussed.

The numerical solutions consist of double power series and they were obtained based on the minimum of the total strain energy [1]. Parametric studies have been performed in order to emphazise the effects of the chosen foundation and that of the geometry.

1. Numerical Solutions of the Thick Plate on Bi-modulus Foundation

Elastic thick plates resting on bi-modulus foundation (Fig. 1) obey the fundamental system of differential equations:

$$(1)\qquad \begin{cases} (K+c_2)\Delta w - c_1 w + K\theta = -p(x,y), \\ \dfrac{D}{2}\left[(1-\nu)\Delta\varphi + \dfrac{\partial\theta}{\partial x} - \nu\dfrac{\partial}{\partial x}\Delta w\right] - K\left(\varphi + \dfrac{\partial w}{\partial x}\right) = 0, \\ \dfrac{D}{2}\left[(1-\nu)\Delta\psi + \dfrac{\partial\theta}{\partial y} - \nu\dfrac{\partial}{\partial y}\Delta w\right] - K\left(\psi + \dfrac{\partial w}{\partial y}\right) = 0, \end{cases}$$

where:

$$K = Gh = \frac{Eh}{2(1+\nu)}, \quad \theta = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y},$$

ν – Poisson's ratio of the plate (usually concrete), D – the plate bending stiffness, Δ – Laplace operator, c_1 and c_2 – the Pasternak (or bi-modulus) model elastic constants, φ and ψ – the complete rotations about x and y axes due to the thin plate bending and due to shear (the thick-Mindlin) model.

When φ and ψ are eliminated, it results [1]:

$$[D(2-\nu)\Delta - 2K]\theta = (\nu D\Delta\Delta + 2K\Delta)w(x,y). \tag{2}$$

Then:

$$D\Delta\Delta w(x,y) = \left(1 - D\frac{2-\nu}{2K}\right)(p - q_s). \tag{3}$$

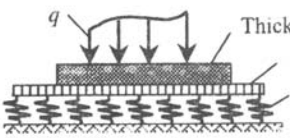


Fig. 1.– Elastic thick plate on bi-modulus foundation.

In Eq. (3) q_s is the response of the elastic bi-modulus foundation:

$$q_s(x,y) = c_1(x,y) - c_2(x,y). \tag{4}$$

From Eqs. (3) and (4) it yields finally:

$$\begin{aligned} D[2K + (2-\nu)c_2]\Delta\Delta w(x,y) - [2Kc_2 + D(2-\nu)c_1]\Delta w(x,y) + 2Kc_1 w = \\ = [2K - D(2-\nu)\Delta]p(x,y). \end{aligned} \tag{5}$$

The additional term of the right member from Eq. (5) takes into account the shear stress and the effect of inter-layer pressure, σ_z.

The approximate solution of the last equation is taken as [1]:

$$w(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}S_{mn}(x,y), \tag{6}$$

and consequently

$$(7\,a)\quad \varphi(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}\frac{\partial}{\partial x}S_{mn}(x,y), \quad (7\,b)\quad \psi(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\frac{\partial}{\partial y}S_{mn}(x,y),$$

where $S_{mn}(x,y) = x^m y^n F(x,y)$ and $F(x,y)$ is a function depending on boundary conditions.

The principle of minimum strain energy was applied. A system of linear equations of infinite degree resulted, which enables to find the coefficients A_{mn}, B_{mn}, C_{mn}.

1.1. Rectangular Hinged Plate Acted by Uniform Load (Fig. 2 *a*)

The load intensity, $p(x,y) = p$, and the boundary function is chosen as:

$$F(x,y) = \prod_{i=1}^{n} [g_i(x,y)]^{p_i}, \tag{8}$$

where $g_i(x,y)$ is the boundary equation and $p_i = 1$ for the simply supported edge. Finally S_{mn} becomes:

$$S_{mn}(x,y) = x^m y^n F(x,y) = x^{m+1} y^{n+1} (x-a)(y-b). \tag{9}$$

The first approximation will be:

$$S_{00} = xy(x-a)(y-b).$$

The complete solution is given in [1].

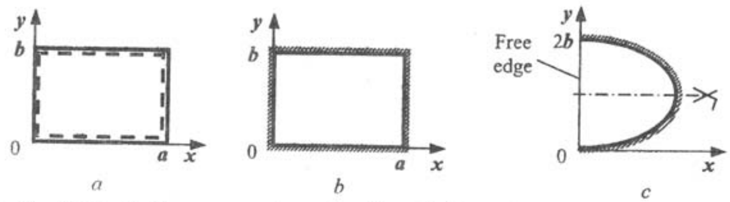


Fig. 2 – Case studies: *a* – rectangular hinged plate; *b* – rectangular clamped plate; *c* – elliptical plate clamped on the curved boundary and free on *y*-axis.

1.2. Rectangular Clamped Plate Acted by a Uniform Load (Fig. 2 *b*)

In this case the boundary function, $F(x,y)$, was chosen as:

$$F(x,y) = x^2(x-a)^2 y^2 (y-b)^2$$

and S_{mn} results from Eq. (9). The fully developed solutions are given in [1], and they require solving 3×3 or 6×6 algebraic systems of linear equation (for the first or the 2nd approximation of the solution). The approximate functions, $w(x,y)$, bending and twisting moments and shear forces depend on the mechanical properties of ground, *i.e.* c_1, which is the Winkler bed coefficient and c_2 is calculated as:

$$c_2 = \frac{H}{3} G_0; \qquad G_0 = \frac{E_0}{2(1+\nu_0)}.$$

E_0 and ν_0 are ground mechanical properties for the plane strain state, but H is the depth of the ground interaction layer [1].

1.3. Semi-elliptical Plate (Fig. 2 *c*)

In this case, the boundary function, $F(x,y)$, was chosen as:

$$F(x,y) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b} \right)^2 \tag{10}$$

and S_{mn} results from Eq. (9):

$$S_{mn} = x^m y^n \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b}\right)^2. \tag{11}$$

The first approximation yields:

$$S_{00} = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b}\right)^2. \tag{12}$$

The vertical displacement, $w(x,y)$, becomes:

$$w_{00}(x,y) = X_1 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b}\right)^2. \tag{13}$$

The rotation angles are:

$$\varphi_{00}(x,y) = \frac{4x}{a^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b}\right) X_2, \tag{14 a}$$

$$\psi_{00}(x,y) = \frac{4(y-b)}{a^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b}\right) X_3. \tag{14 b}$$

The bending and twisting moments are:

$$M_x(x,y) = -2D\left[\frac{A}{a^2}B(x,y) + \nu\frac{C}{b^2}R(x,y)\right], \tag{15 a}$$

$$M_y(x,y) = -2D\left[\frac{C}{a^2}R(x,y) + \nu\frac{A}{b^2}B(x,y)\right], \tag{15 b}$$

$$M_{xy}(x,y) = -2D(1-\nu)\frac{x(y-b)}{a^2b^2}(X_1 + X_2), \tag{15 c}$$

where

$$A = (2-\nu)X_2 - \nu X_1, \tag{16 a}$$ $$C = X_3 - X_1, \tag{16 b}$$

$$B(x,y) = \frac{3x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b}, \tag{16 c}$$ $$R(x,y) = \frac{x^2}{a^2} + \frac{3y^2}{b^2} - \frac{6y}{b} + 2. \tag{16 d}$$

$X_1,\ldots,X_3$ are the solutions of the matrix equation:

$$\mathbf{AX} = \delta. \tag{17}$$

The matrices $\mathbf{A}$ and δ are

$$\mathbf{A} = \begin{bmatrix} A_1 & b^2A_2 & a^2A_3 \\ A_2 & A_4 & a^2A_5 \\ A_3 & b^2A_5 & A_6 \end{bmatrix}, \qquad \delta = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} \tag{18}$$

and depend on plate mechanical properties and geometry and also on bi-modulus subgrade mechanical properties.

Here

$$\begin{aligned} A_1 = {} & \frac{\nu^2}{1+\nu}\cdot\frac{7}{6\cdot 17}\left[45\left(a^4+b^4\right)+29a^2b^2\right]+\alpha\left(a^2+b^2\right)+ \\ & +\frac{c_1}{D}\cdot\frac{263a^4b^4}{15\cdot 16\cdot 17}+\frac{c_2}{D}\cdot\frac{a^2b^2\left(23a^2+257b^2\right)}{12\cdot 17}, \end{aligned} \tag{19a}$$

$$A_2 = -\frac{\nu(1-\nu)}{1+\nu}\cdot\frac{105}{4\cdot 17}\left(29a^2+45b^2\right)+\alpha, \tag{19b}$$

(19 c) $A_3 = -\dfrac{\nu(1-\nu)}{1+\nu}\cdot\dfrac{105}{4\cdot 17}\left(29b^2+45a^2\right)+\alpha,$ (19 d) $A_4 = \beta b^2+\beta_1 a^2+\alpha,$

(19 e) $A_5 = \dfrac{7}{6\cdot 17}\cdot\dfrac{19\nu^2+10}{1+\nu},$ (19 f) $A_6 = \beta a^2+\beta_1 b^2+\alpha,$

$$\alpha = \frac{6(1-\nu)}{5h^2}, \quad \beta = \frac{1-\nu-\nu^2(1+\nu^2)}{1-\nu^2}\cdot\frac{105}{2\cdot 17}, \quad \beta_1 = 35\frac{(1-\nu)}{3\cdot 17}, \tag{20a}$$

$$\delta_1 = p\left[\frac{7a^4b^4}{8\cdot 24D}-\frac{105}{8\cdot 17}\cdot\frac{\nu^2}{Eh}a^2b^2(a^2+b^2)\right], \ \delta_2 = -p\frac{105}{8\cdot 17}\cdot\frac{\nu^2}{Eh}a^2b^2, \ \delta_3 = \delta_2. \tag{20b}$$

2. Models for Finite Element Analysis

Modelling is an art based on the ability to visualise physical interactions. All basic and applied knowledge of physical problems, finite elements and solution algorithms contribute to modelling expertise.

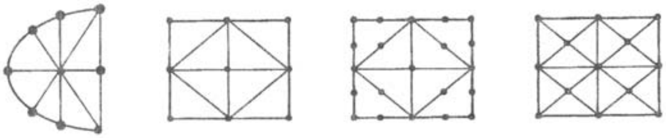
Sometimes, the user of a computer program does not understand the physical action and boundary conditions of the actual structure, and the limitations of applicable theory, well enough to prepare a satisfactory model.

Another difficulty occurs when the behaviour of various elements, the program's limitations and options were not understood well enough to make an intelligent choice among them. In this case the model fails to reflect important feature of the physical problem, and sometimes there is obtained an overflow of computed results, which are not properly examined and questioned.

Choosing one element-type or mesh or others, is another problem. The analyst must understand how various elements behave in various situations. The practice of using Finite Element-based software showed that elements and meshes of intermediate complexity are better fitted to many problems.

A coarse mesh may not always depict the actual structure. Figs. 3 *a* and 3 *b* show thick plates, clamped on the boundary, modelled by coarse meshes. The model surface is much smaller than the actual one, since there is only one internal node, which is not restrained. In this case, the output results will be based only on nodal parameters of the non-restrained node.

Even the selection of the same number of elements but refined, with nodes at the mid-edge helps to a better approximation (Fig. 3 *c*). Increasing the number of elements (Fig. 3 *d*) leads to a similar result, since there are five free-to-move nodes in the last two cases.



a *b* *c* *d*

Fig. 3.– Coarse – mesh alternatives of modelling a plate clamped-on the boundary.

The model will comprise now all structural parts, including those carrying little load or little stress. Most of commercial software may model curved boundary, but when this option is not available, the substituting polygon or polyhedron must preserve the actual volume of the structure. When the mesh is irregular, the recommended angles are from 45° to 135°.

At least two layers of solid (brick) elements must be used, to obtain the stress from the median surface.

Solid three-dimensional element [4] does not allow rotations, so that in this case the boundary conditions for simply supported and hinged edges are not accurately depicted. Additional elements like double hinged rigid rods may help to allow the edge rotation, if the software capabilities do not include thick plate elements.

3. Parametric Studies

Parametric studies were performed in order to compare, for each type of previously presented plate, the numerical solution to other solutions and also to FEA results.

Winkler's classical case has been obtained, too, by setting $c_2 = 0$.

The clamped and hinged plates were quadratic, and the variable parameters were the ratio of Young moduli plate-subgrade, and the influence depth, H, was also modified, for one case of analysis.

For elliptical plate, the ratios h/a and h/b were modified also, to state bounds of validity for the thick plate theory.

The obtained results are presented in Table 1, and the next Section refers to them.

Table 1
A Synthesis of Parametric Studies

$p = 10^7$ N/m^2, square plate, $a = 30$ m, $h = a/10$	Vertical deflection, [mm], at the plate centre					
Simple supported plate						
$n = E_{plate}/E_{soil}$	10^3	10^2			10^1	10^0
$H = 12$ m	H	H	$2H$	$3H$	H	H
Polynomial series, one term	142.412	6.887	63.683	61.623	38.019	11.857
Polynomial series, two terms	145.015	66.438	64.199	62.105	38.201	11.875
Winkler model, polynomial series, one term	143.408	68.248			47.501	36.428
Winkler model, polynomial series, two terms	146.048	68.843			47.708	36.596
FEA	65.45	65.22	60.797	60.522	47.8846	15.0136
Clamped plate						
Polynomial series, one term	20.4128	17.546	17.201	16.87	13.683	5.2274
Winkler model, polynomial series, one term	20.4572	17.906			16.22	14.824
Semi-elliptical plate: $a = 12$ m, $b = 9$ m, $h = 3$ m, $H = 12$ m – Maximum vertical deflection						
Polynomial series, one term	3.05627	3.15555	3.054985	2.96067	2.3228	0.68687
Winkler model, polynomial series, one term	3.065358	3.26446			3.0653	2.599
FEA (coarse mesh)	10.703	10.391			8.099	3.093

4. Conclusions

1. *Winkler's Model versus Bi-modulus Model.* Winkler's model behaves much more elastically, but this feature is more obvious for $10^0 < E_{plate}/E_{soil} < 10^2$. The clay subgrade ($E_{plate}/E_{soil} = 10^3$) is much more deformable than other cases. In this case, for rectangular plates the results were of the same order of magnitude, both for bi-modulus and for Winkler subgrade. For the semi-elliptical plate the results are different for $10^0 < E_{plate}/E_{soil} < 10^1$.

2. *Influence of Interaction Layer Depth, H.* The depth of the interaction layer has a little influence, less than 6% (Table 1) for almost all-approximate solutions for all cases.

3. *Influence of Ratio E_{plate}/E_{soil}.* It is obviously that the subgrade mechanical properties would influence the deflections, and then the reactive pressures, and the

distribution of internal forces. This effect is more important for high ratios, for all analysed plates.

4. *Influence of Aspect Ratio h/a for Semi-elliptical Plate.* Fig. 4 and Table 2 show the importance of this ratio; otherwise, applying Kirchhoff theory instead of the thick plate one would cause significant errors. The higher the aspect ratio, the more flexible becomes the plate. Further studies are required to state accurate bounds of using one or other theory (or one or the other type of finite elements). The effect depends also on the general plate shape.

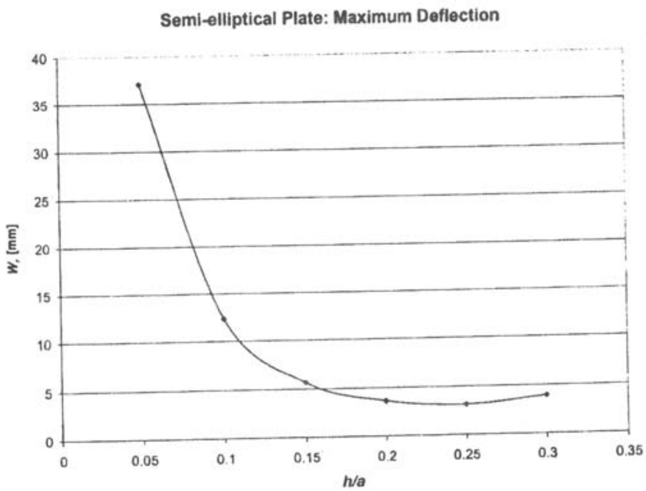


Fig. 4.– Plot of vertical deflection in terms of h/a ratio.

Table 2
Semi-elliptical Plate: Bending Moment Distribution in Terms of h/a Ratio

h/a $E_{\text{plate}}/E_{\text{soil}} = 10^2$	M_x $(\text{N}\cdot\text{m})\cdot 10^{-5}$	M_y $(\text{N}\cdot\text{m})\cdot 10^{-5}$
0.05	8.436	1.761
0.10	17.022	4.091
0.15	16.548	5.233
0.20	12.965	6.644
0.25	8.838	9.747
0.30	12.871	30.326

5. *FEA versus Polynomial Approximation.* FEA analysis has been performed by SAP2000 program. "Brick" 8-node elements were chosen, but these elements cannot represent the plate-bending phenomenon, since all rotations are restrained. Satisfactory and close results were obtained for small $E_{\text{plate}}/E_{\text{soil}}$ aspect ratios, especially for rectangular plates. A very coarse mesh has been used to model the semi-elliptical plate, where the results are of another order of magnitude. For further studies it is

recommended to build up an AutoCAD model and then import it for analysis.

Received, July 26, 2004

Technical University "Gh.Asachi", Jassy,
Department of Hydrotechnical Structures
and
**Department of Structural Mechanics*

REFERENCES

1. Manole O., *Plates on Elastic Foundation. Methods for Solving Thick Plates on Elastic Foundation* (in Romanian). Ph. D. Diss., Polyt. Inst. of Cluj-Napoca, Romania, 1989.
2. Cook R., Malkus D., Plesha M., *Concepts and Applications of Finite Element Analysis.* J. Wiley & Sons, NY, 1989.
3. Hughes T.J.R., Tezduyar T.E., *Finite Elements Based on Mindlin Plate Theory with Particular Reference to the Four-Node Bilinear Isoparametric Element.* J. Appl. Mech., **48**, *3*, 587-596 (1981).
4. * * * www.csiberkeley.com. SAP 2000 Software User's Manual.
5. Chandrupatla T., Belegundu A., *Introduction to Finite Elements in Engineering.* 3rd Ed., Pearson Educ., Indian Branch, Delhi, India, 2002.

STUDIU COMPARATIV AL SOLUŢIILOR PENTRU PLĂCILE ELASTICE GROASE REZEMATE PE FUNDAŢIE BIMODUL

(Rezumat)

Se analizează, comparativ, soluţiile obţinute pentru încovoierea plăcilor groase rezemate pe mediu elastic cu două caracteristici, incluzând efectul forfecării şi al efortului normal. Soluţiile au fost obţinute prin metoda elementelor finite şi prin metode numerice bazate pe minimizarea energiei totale. S-au analizat diferite condiţii de contur geometrice (placa dreptunghiulară, semi-eliptică) şi de rezemare (încastrare, simplă rezemare).