A COMPARATIVE STUDY OF SOLUTIONS CONCERNING THICK ELASTIC PLATES ON BI-MODULUS FOUNDATION

BY

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The classical bending theory of elastic plates is based upon the assumption that the internal moments are proportional to the curvatures of the median deformed surface. This theory does not include the effects of shear and normal pressure in the plate. The model of a bi-modulus foundation is a realistic generalization of the Winkler's classical one and is widely used to represent the subgrade of railroad systems, airport lanes [1], [2]. The derived equation of elastic thick plates on bi-modulus foundation considers shear and normal stress as linear variable across the plate thickness.

This paper presents numerical solutions for thick plate resting on bi-modulus subgrade. These solutions take into account the shear distortion, and they are compared to the solutions obtained by Finite Element Analysis and with the Winkler's model. Particular solutions for the rectangular plate of clamped boundary, for the hinged rectangular plate and for a semi-elliptical plate, are discussed.

The numerical solutions consist of double power series and they were obtained based on the minimum of the total strain energy [1]. Parametric studies have been performed in order to emphasize the effects of the chosen foundation and that of the geometry.

1. Numerical Solutions of the Thick Plate on Bi-modulus Foundation

Elastic thick plates resting on bi-modulus foundation (Fig. 1) obey the fundamental system of differential equations:

\[
\begin{align*}
(K + c_2)\Delta w - c_1 w + K\theta &= -p(x, y), \\
\frac{D}{2} \left[ (1 - \nu)\Delta \varphi + \frac{\partial \varphi}{\partial x} - \nu \frac{\partial}{\partial x} \Delta w \right] - K \left( \varphi + \frac{\partial w}{\partial x} \right) &= 0, \\
\frac{D}{2} \left[ (1 - \nu)\Delta \psi + \frac{\partial \psi}{\partial y} - \nu \frac{\partial}{\partial y} \Delta w \right] - K \left( \psi + \frac{\partial w}{\partial y} \right) &= 0,
\end{align*}
\]

where:

\[
K = Gh = \frac{Eh}{2(1 + \nu)}, \quad \theta = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y},
\]

\(\nu\) – Poisson’s ratio of the plate (usually concrete), \(D\) – the plate bending stiffness, \(\Delta\) – Laplace operator, \(c_1\) and \(c_2\) – the Pasternak (or bi-modulus) model elastic
constants, \( \varphi \) and \( \psi \) – the complete rotations about \( x \) and \( y \) axes due to the thin plate bending and due to shear (the thick Mindlin) model.

When \( \varphi \) and \( \psi \) are eliminated, it results [1]:

\[
[D(2 - \nu)\Delta - 2K]\theta = (\nu D\Delta\Delta + 2K\Delta)w(x, y).
\]

Then:

\[
D\Delta\Delta w(x, y) = \left(1 - D\frac{2 - \nu}{2K}\right)(p - q_s).
\]

Fig. 1 - Elastic thick plate on bi-modulus foundation.

In Eq. (3) \( q_s \) is the response of the elastic bi-modulus foundation:

\[
q_s(x, y) = c_1(x, y) - c_2(x, y).
\]

From Eqs. (3) and (4) it yields finally:

\[
D[2K + (2 - \nu)c_2]\Delta\Delta w(x, y) - [2Kc_2 + D(2 - \nu)c_1]\Delta w(x, y) + 2Kc_1w = \]

\[
= [2K - D(2 - \nu)\Delta]p(x, y).
\]

The additional term of the right member from Eq. (5) takes into account the shear stress and the effect of inter-layer pressure, \( \sigma_z \).

The approximate solution of the last equation is taken as [1]:

\[
w(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} S_{mn}(x, y),
\]

and consequently:

\[
(7a) \quad \varphi(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} \frac{\partial}{\partial x} S_{mn}(x, y), \quad (7b) \quad \psi(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} \frac{\partial}{\partial y} S_{mn}(x, y),
\]

where \( S_{mn}(x, y) = x^m y^n F(x, y) \) and \( F(x, y) \) is a function depending on boundary conditions.

The principle of minimum strain energy was applied. A system of linear equations of infinite degree resulted, which enables to find the coefficients \( A_{mn}, B_{mn}, C_{mn} \).
1.1. Rectangular Hinged Plate Acted by Uniform Load (Fig. 2a)

The load intensity, \( p(x, y) = p \), and the boundary function is chosen as:

\[
F(x, y) = \prod_{i=1}^{n} [g_i(x, y)]^{p_i},
\]

where \( g_i(x, y) \) is the boundary equation and \( p_i = 1 \) for the simply supported edge. Finally \( S_{mn} \) becomes:

\[
S_{mn}(x, y) = x^m y^n F(x, y) = x^{m+1} y^{n+1} (x - a)(y - b).
\]

The first approximation will be:

\[
S_{00} = xy(x - a)(y - b).
\]

The complete solution is given in [1].

![Case studies: a - rectangular hinged plate; b - rectangular clamped plate; c - elliptical plate clamped on the curved boundary and free on y-axis.](image)

1.2. Rectangular Clamped Plate Acted by a Uniform Load (Fig. 2b)

In this case the boundary function, \( F(x, y) \), was chosen as:

\[
F(x, y) = x^2(x - a)^2 y^2(y - b)^2
\]

and \( S_{mn} \) results from Eq. (9). The fully developed solutions are given in [1], and they require solving \( 3 \times 3 \) or \( 6 \times 6 \) algebraic systems of linear equation (for the first or the \( 2^{nd} \) approximation of the solution). The approximate functions, \( w(x, y) \), bending and twisting moments and shear forces depend on the mechanical properties of ground, i.e. \( c_1 \), which is the Winkler bed coefficient and \( c_2 \) is calculated as:

\[
c_2 = \frac{H}{3} G_0; \quad G_0 = \frac{E_0}{2(1 + \nu)}.
\]

\( E_0 \) and \( \nu_0 \) are ground mechanical properties for the plane strain state, but \( H \) is the depth of the ground interaction layer [1].

1.3. Semi-elliptical Plate (Fig. 2c)

In this case, the boundary function, \( F(x, y) \), was chosen as:

\[
F(x, y) = \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b} \right)^2
\]
and $S_{mn}$ results from Eq. (9):

\begin{equation}
S_{mn} = x^m y^n \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b} \right)^2.
\end{equation}

The first approximation yields:

\begin{equation}
S_{00} = \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b} \right)^2.
\end{equation}

The vertical displacement, $w(x, y)$, becomes:

\begin{equation}
w_{00}(x, y) = X_1 \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b} \right)^2.
\end{equation}

The rotation angles are:

\begin{equation}
\varphi_{00}(x, y) = \frac{4x}{a^2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b} \right) X_2,
\end{equation}

\begin{equation}
\psi_{00}(x, y) = \frac{4(y - b)}{a^2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b} \right) X_3.
\end{equation}

The bending and twisting moments are:

\begin{equation}
M_x(x, y) = -2D \left[ \frac{A}{a^2} B(x, y) + \frac{C}{b^2} R(x, y) \right],
\end{equation}

\begin{equation}
M_y(x, y) = -2D \left[ \frac{C}{a^2} R(x, y) + \frac{A}{b^2} B(x, y) \right],
\end{equation}

\begin{equation}
M_{xy}(x, y) = -2D(1 - \nu) \frac{x(y - b)}{a^2 b^2} (X_1 + X_2),
\end{equation}

where

\begin{equation}
A = (2 - \nu) X_2 - \nu X_1, \quad (16b) \quad C = X_3 - X_1,
\end{equation}

\begin{equation}
B(x, y) = \frac{3x^2}{a^2} + \frac{y^2}{b^2} - \frac{2y}{b}, \quad (16d) \quad R(x, y) = \frac{x^2}{a^2} + \frac{3y^2}{b^2} - \frac{6y}{b} + 2.
\end{equation}

$X_1, \ldots, X_3$ are the solutions of the matrix equation:

\begin{equation}
A X = \delta.
\end{equation}
The matrices $A$ and $\delta$ are

$$
A = \begin{bmatrix}
A_1 & b^2A_2 & a^2A_3 \\
A_2 & A_4 & a^2A_5 \\
A_3 & b^2A_5 & A_6
\end{bmatrix}
$$

and depend on plate mechanical properties and geometry and also on bi-modulus subgrade mechanical properties.

Here

$$
A_1 = \frac{\nu^2}{1+\nu} \cdot \frac{7}{6 \cdot 17} \left[ 45 \left( a^4 + b^4 \right) + 29a^2b^2 \right] + \alpha \left( a^2 + b^2 \right) + \frac{c_1}{D} \cdot \frac{263}{15 \cdot 16 \cdot 17} a^4 b^4 + \frac{c_2}{D} \cdot \frac{a^2b^2 \left( 23a^2 + 257b^2 \right)}{12 \cdot 17},
$$

(19a)

$$
A_2 = -\frac{\nu(1-\nu)}{1+\nu} \cdot \frac{105}{4 \cdot 17} \left( 29a^2 + 45b^2 \right) + \alpha,
$$

(19b)

$$
A_3 = -\frac{\nu(1-\nu)}{1+\nu} \cdot \frac{105}{4 \cdot 17} \left( 29b^2 + 45a^2 \right) + \alpha,
$$

(19c)

$$
A_4 = \beta b^2 + \beta_1 a^2 + \alpha,
$$

(19d)

$$
A_5 = \frac{7}{6 \cdot 17} \cdot \frac{19\nu^2 + 10}{1+\nu},
$$

(19e)

$$
A_6 = \beta a^2 + \beta_1 b^2 + \alpha,
$$

(19f)

$$
\alpha = \frac{6(1-\nu)}{5h^2}, \quad \beta = \frac{1 - \nu - \nu^2(1+\nu^2)}{1-\nu^2} \cdot \frac{105}{2 \cdot 17}, \quad \beta_1 = \frac{35(1-\nu)}{3 \cdot 17},
$$

(20a)

$$
\delta_1 = p \left[ \frac{7a^4b^4}{8 \cdot 24D} - \frac{105}{8 \cdot 17} \cdot \frac{\nu^2}{Eh} a^2b^2(a^2 + b^2) \right], \quad \delta_2 = -p \frac{105}{8 \cdot 17} \cdot \frac{\nu^2}{Eh} a^2b^2, \quad \delta_3 = \delta_2.
$$

(20b)

2. Models for Finite Element Analysis

Modelling is an art based on the ability to visualise physical interactions. All basic and applied knowledge of physical problems, finite elements and solution algorithms contribute to modelling expertise.

Sometimes, the user of a computer program does not understand the physical action and boundary conditions of the actual structure, and the limitations of applicable theory, well enough to prepare a satisfactory model.
Another difficulty occurs when the behaviour of various elements, the program's limitations and options were not understood well enough to make an intelligent choice among them. In this case the model fails to reflect important feature of the physical problem, and sometimes there is obtained an overflow of computed results, which are not properly examined and questioned.

Choosing one element-type or mesh or others, is another problem. The analyst must understand how various elements behave in various situations. The practice of using Finite Element-based software showed that elements and meshes of intermediate complexity are better fitted to many problems.

A coarse mesh may not always depict the actual structure. Figs. 3a and 3b show thick plates, clamped on the boundary, modelled by coarse meshes. The model surface is much smaller than the actual one, since there is only one internal node, which is not restrained. In this case, the output results will be based only on nodal parameters of the non-restrained node.

Even the selection of the same number of elements but refined, with nodes at the mid-edge helps to a better approximation (Fig. 3c). Increasing the number of elements (Fig. 3d) leads to a similar result, since there are five free-to-move nodes in the last two cases.

![Fig. 3. - Coarse mesh alternatives of modelling a plate clamped-on the boundary](image)

The model will comprise now all structural parts, including those carrying little load or little stress. Most of commercial software may model curved boundary, but when this option is not available, the substituting polygon or polyhedron must preserve the actual volume of the structure. When the mesh is irregular, the recommended angles are from $45^\circ$ to $135^\circ$.

At least two layers of solid (brick) elements must be used, to obtain the stress from the median surface.

Solid three-dimensional element [4] does not allow rotations, so that in this case the boundary conditions for simply supported and hinged edges are not accurately depicted. Additional elements like double hinged rigid rods may help to allow the edge rotation, if the software capabilities do not include thick plate elements.

3. Parametric Studies

Parametric studies were performed in order to compare, for each type of previously presented plate, the numerical solution to other solutions and also to FEA results.
Winkler’s classical case has been obtained, too, by setting \( c_2 = 0 \).

The clamped and hinged plates were quadratic, and the variable parameters were the ratio of Young moduli plate-subgrade, and the influence depth, \( H \), was also modified, for one case of analysis.

For elliptical plate, the ratios \( h/a \) and \( h/b \) were modified also, to state bounds of validity for the thick plate theory.

The obtained results are presented in Table 1, and the next Section refers to them.

### Table 1

*A Synthesis of Parametric Studies*

<table>
<thead>
<tr>
<th>( n = E_{\text{plate}}/E_{\text{soil}} )</th>
<th>Vertical deflection, [mm], at the plate centre</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = 12 \text{ m} )</td>
<td></td>
</tr>
<tr>
<td>Polynomial series, one term</td>
<td>( 10^3 ) ( 10^2 ) ( 10^1 ) ( 10^0 )</td>
</tr>
<tr>
<td>( H )</td>
<td>( H ) ( 2H ) ( 3H ) ( H ) ( H )</td>
</tr>
<tr>
<td>Winkler model, polynomial series, one term</td>
<td>142.412 6.887 63.683 61.623 38.019 11.857</td>
</tr>
<tr>
<td>Winkler model, polynomial series, two terms</td>
<td>145.015 66.438 64.199 62.105 38.201 11.875</td>
</tr>
<tr>
<td>FEA</td>
<td>65.45 65.22 60.797 60.522 47.846 15.0136</td>
</tr>
<tr>
<td>Polynomial series, one term</td>
<td></td>
</tr>
<tr>
<td>Winkler model, polynomial series, one term</td>
<td>20.4128 17.546 17.201 16.87 13.683 5.2274</td>
</tr>
<tr>
<td>Winkler model, polynomial series, two terms</td>
<td>20.6572 17.906 16.22 14.824</td>
</tr>
<tr>
<td>Semi-elliptical plate: ( a = 12 \text{ m}, \ b = 9 \text{ m}, \ h = 3 \text{ m}, \ H = 12 \text{ m} )</td>
<td>Maximum vertical deflection</td>
</tr>
<tr>
<td>Polynomial series, one term</td>
<td></td>
</tr>
<tr>
<td>Winkler model, polynomial series, one term</td>
<td>3.05627 3.15555 3.054986 2.96067 2.3228 0.68887</td>
</tr>
<tr>
<td>FEA (coarse mesh)</td>
<td></td>
</tr>
</tbody>
</table>

4. Conclusions

1. **Winkler’s Model versus Bi-modulus Model.** Winkler’s model behaves much more elastically, but this feature is more obvious for \( 10^0 < E_{\text{plate}}/E_{\text{soil}} < 10^2 \). The clay subgrade \( (E_{\text{plate}}/E_{\text{soil}} = 10^8) \) is much more deformable than other cases. In this case, for rectangular plates the results were of the same order of magnitude, both for bi-modulus and for Winkler subgrade. For the semi-elliptical plate the results are different for \( 10^0 < E_{\text{plate}}/E_{\text{soil}} < 10^1 \).

2. **Influence of Interaction Layer Depth, \( H \).** The depth of the interaction layer has a little influence, less than 6% (Table 1) for almost all-approximate solutions for all cases.

3. **Influence of Ratio \( E_{\text{plate}}/E_{\text{soil}} \).** It is obviously that the subgrade mechanical properties would influence the deflections, and then the reactive pressures, and the
distribution of internal forces. This effect is more important for high ratios, for all analysed plates.

4. Influence of Aspect Ratio $h/a$ for Semi-elliptical Plate. Fig. 4 and Table 2 show the importance of this ratio; otherwise, applying Kirchhoff theory instead of the thick plate one would cause significant errors. The higher the aspect ratio, the more flexible becomes the plate. Further studies are required to state accurate bounds of using one or other theory (or one or the other type of finite elements). The effect depends also on the general plate shape.

![Semi-elliptical Plate: Maximum Deflection](image)

Fig. 4. Plot of vertical deflection in terms of $h/a$ ratio.

<table>
<thead>
<tr>
<th>$h/a$ $E_{\text{plate}}/E_{\text{soil}} = 10^2$</th>
<th>$M_x$ $(\text{N}\cdot\text{m}) \cdot 10^{-5}$</th>
<th>$M_y$ $(\text{N}\cdot\text{m}) \cdot 10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>8.436</td>
<td>1.761</td>
</tr>
<tr>
<td>0.10</td>
<td>17.022</td>
<td>4.091</td>
</tr>
<tr>
<td>0.15</td>
<td>16.548</td>
<td>5.233</td>
</tr>
<tr>
<td>0.20</td>
<td>12.965</td>
<td>6.644</td>
</tr>
<tr>
<td>0.25</td>
<td>8.838</td>
<td>9.747</td>
</tr>
<tr>
<td>0.30</td>
<td>12.871</td>
<td>30.326</td>
</tr>
</tbody>
</table>

5. FEA versus Polynomial Approximation. FEA analysis has been performed by SAP2000 program. "Brick" 8-node elements were chosen, but these elements cannot represent the plate-bending phenomenon, since all rotations are restrained. Satisfactory and close results were obtained for small $E_{\text{plate}}/E_{\text{soil}}$ aspect ratios, especially for rectangular plates. A very coarse mesh has been used to model the semi-elliptical plate, where the results are of another order of magnitude. For further studies it is
recommended to build up an AutoCAD model and then import it for analysis.

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REFERENCES


STUDIU COMPARATIV AL SOLUȚIILOR PENTRU PLĂCILE ELASTICE GROASE REZEMATE PE FUNDĂTIE BIMODUL

(Rezumat)

Se analizează, comparativ, soluțiile obținute pentru înconvoierea plăcilor groase rezemate pe un mediu elastic cu două caracteristici, inclusiv efectul forfecării și al efortului normal. Soluțiile au fost obținute prin metoda elementelor finite și prin metode numerice bazate pe minimizarea energiei totale. S-au analizat diferite condiții de contur geometrice (placa dreptunghiulară, semicirculară) și de rezemare (încastrare, simplă rezemare).