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GALERKIN MESHLESS FORMULATIONS FOR 3D BEAM PROBLEMS

BY

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Abstract. The main idea of meshless methods is to approximate the unknown field by a linear combination of shape functions built without having recourse to a mesh of the domain. The computational domain is discretized using a set of scattered nodes. The shape functions associated with a given node is then built considering the weight functions whose support overlaps the one of the weight function of this node; thus, there is actually no need to establish connectivities between the different nodes as in the finite element method. Monte-Carlo integration techniques are promising schemes in the context of meshless techniques. The purpose of the present paper is to implement in EFG a new body integration technique for the evaluation of the stiffness matrix that does not rely on a partition of the domain into cells, but rather points. Numerical examples based on three-dimensional elasticity problems are presented to examine the accuracy and convergence of the proposed method. In this context, Quasi-Monte Carlo integration techniques are used. The results are compared to traditional EFG. Conclusions are drawn concerning the proposed techniques and its capabilities.
Key words: Meshless formulations, EFG, 3D elasticity, Monte-Carlo integration techniques

1. Introduction

In engineering one often has a number of data points, as obtained by sampling or some experiment, and tries to construct a function, which closely fits those data points. The so-called meshless methods construct approximations from a set of nodal data without the need for any (finite - element) *a priori* connectivity information between the nodes. In general, a meshless method uses a local interpolation or approximation to represent the trial function with the values (or the fictitious values) of the unknown variable at some randomly located nodes.

The fast convergence, ease of adaptive refinement, trivial rising of the consistency order and the continuity of derivatives up to the desired order are features of this class of methods.

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The meshless approximations functions constructed in continuous or in discrete are used to approximate the displacement (or other variables of interest) to solve applied mechanics problems.

The approximations are used as approximations of the strong forms of partial differential equations (PDEs), and those serving as approximations of the weak forms of PDEs to set up a linear system of equations. To approximate the strong form of a PDE using a particle method, the differential equation with partial derivatives is usually discretized by a specific collocation technique. To approximate the weak form of a PDE various Galerkin weak formulations are used. Several types of discrete approximation functions can be used; among these can be found: a) moving least square (MLS) functions, b) partition of unity (PU) functions, or *hp*-cloud functions, as representatives. Surveys can be found in [6].

2. Discrete Moving Least Squares: EFG interpolation

The basic idea of the MLS approach is to approximate $u(\mathbf{x})$, at a given point $\boldsymbol{x}$, through a polynomial least-squares fitting of u in a neighborhood of $\mathbf{x}$. That is, $u(\mathbf{x})$ is approximated with the polynomical expression

(1) $$u(\mathbf{x}) \approx u^h(\mathbf{x}) = \sum_j^m p_j(\mathbf{x}) a_j = \boldsymbol{p}^T(\mathbf{x})\boldsymbol{a}(\mathbf{x})$$

where $\boldsymbol{p}^T(\boldsymbol{x}) = [\ \mathbf{p}_1(\boldsymbol{x}),\ \ \mathbf{p}_2(\boldsymbol{x}),\ \ldots,\ \ \mathbf{p}_m(\boldsymbol{x})\ ]$ is a vector of complete basis functions of order m. In the framework of the Element Free Galerkin method, the vector $\boldsymbol{a}(\boldsymbol{x})$ is obtained through a least-squares fitting, by means of minimizing the square of the distance between n data values defined at the points $\boldsymbol{x}_i$ and an approximating function evaluated at the same points

(2) $$J(\boldsymbol{x}) = \sum_{i=1}^N w_i \left(u^h(\mathbf{x}_i) - \mathbf{u}_i\right)^2 = \sum_{i=1}^N w_i \left(\boldsymbol{p}^T(\mathbf{x}_i)\boldsymbol{a} - \mathbf{u}_i\right)^2 = \min$$

That is, $\boldsymbol{a}(\boldsymbol{x})$ is the solution of the linear system of equations given by eq. (2):

(3) $$\boldsymbol{a}(\boldsymbol{x}) = \mathbf{A}^{-1}(\boldsymbol{x})\mathbf{B}(\boldsymbol{x})\hat{\mathbf{u}}$$

A number of weight functions are available in the literature ([2],…, [5]). In this paper, we use cubic spline weight functions with circular domain

(4) $$w(r) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3 & \text{for } r \le \frac{r_I}{2} \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3 & \text{for } \frac{r_I}{2} < r \le r_I \\ 0 & \text{for } r > r_I \end{cases}$$

where $r = \|\mathbf{x} - \mathbf{x}_I\|$ is the distance from the chosen point and r_I is the size of the support for the weight function of center I.

After substitution of the solution of eq. (3) in eq. (1), the least-squares approximation of u in a neighborhood of $\mathbf{x}$ is obtained

$$u^h(\mathbf{x}) = \mathbf{\Phi}^{\mathbf{T}}(\mathbf{x}) \cdot \hat{\mathbf{u}} = \sum_{i=1}^{N} \phi_i(\mathbf{x}) \hat{u}_i \tag{5}$$

where

$$\phi_i(\mathbf{x}) = \boldsymbol{p}^{\mathbf{T}}(\mathbf{x}) \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}_i(\mathbf{x}) \tag{6}$$

The $\boldsymbol{\phi}_i(\mathbf{x})$ are called the *interpolation functions of the MLS approximation*.

In matrix form, the arrays $\mathbf{A}(\boldsymbol{x})$ and $\mathbf{B}(\boldsymbol{x})$ may be written as

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \mathbf{P}^T \mathbf{W}(\|\Delta\mathbf{x}\|) \mathbf{P} \\ \mathbf{B}(\mathbf{x}) &= \mathbf{W}(\|\Delta\mathbf{x}\|) \mathbf{P} \end{aligned} \tag{7}$$

where

$$\mathbf{W}(\|\Delta\mathbf{x}\|) = \begin{bmatrix} w_1(\|\mathbf{x}-\mathbf{x}_1\|) & 0 & \cdots & \cdots \\ 0 & w_2(\|\mathbf{x}-\mathbf{x}_2\|) & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \cdots & \cdots & 0 & w_N(\|\mathbf{x}-\mathbf{x}_N\|) \end{bmatrix} \tag{8}$$

and

$$\mathbf{P} = \begin{bmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \cdots & p_{\mathrm{m}}(\mathbf{x}_1) \\ \vdots & \vdots & \vdots & \vdots \\ p_1(\mathbf{x}_1) & p_2(\mathbf{x}_N) & \cdots & p_{\mathrm{m}}(\mathbf{x}_N) \end{bmatrix} \tag{9}$$

3. Meshless Based on a Galerkin Weak Form

The mesh-free shape functions can also be used in the discretization of the weak integral form of the boundary value problem. For small displacements in three-dimensional, isotropic and linear elastic solids, the equilibrium equation and the boundary conditions are

$$\Delta\boldsymbol{\sigma} + \mathbf{b} = 0 \text{ in } \Omega \tag{10}$$

and

$$\begin{cases} \mathbf{N}\boldsymbol{\sigma} = \overline{\mathbf{t}} & \text{on } \Gamma_{\mathrm{t}} \text{ (natural boundary} \\ \boldsymbol{u} = \overline{\mathbf{u}} & \text{on } \Gamma_{\mathrm{u}} \text{ (essential boundary conditions)} \end{cases} \tag{11}$$

where $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ is the stress vector, **D** – the material property matrix, $\boldsymbol{\varepsilon} = \nabla^s \boldsymbol{u}$ – the strain vector, **u** – the displacement field, **b** – a body force vector, $\overline{\mathbf{t}}$ is the prescribed traction vector on Neumann boundary, Γ_t, and $\overline{\mathbf{u}}$ – the vector of prescribed displacements on Dirichlet boundary, Γ_u. $\Delta^T = \{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$ is the vector of gradient operators, $\nabla^s \boldsymbol{u}$ is the symmetric part of $\nabla \boldsymbol{u}$.

The variational form of eqs. (10) and (11) is given by:

$$\int_\Omega \boldsymbol{\sigma}^T \delta\boldsymbol{\varepsilon} d\Omega - \int_\Omega \boldsymbol{b}^T \delta\boldsymbol{u} d\Omega - \int_{\Gamma_t} \overline{\boldsymbol{t}}^T \delta \mathrm{u} d\Gamma - \delta W_u = 0 \tag{12}$$

where δ denotes the variation operator and δW_u represents a term that enforces essential boundary conditions. The explicit form of this term depends on the method by which the essential boundary conditions are imposed ([1],…,[3]). In this study, W_u is finding using Lagrange multipliers and is evaluated using the following relation:

$$\delta W_u = \int_{\Gamma_u} \delta\lambda(\mathrm{u} - \overline{\mathbf{u}}) d\Gamma + \int_{\Gamma_u} \lambda \delta \mathrm{u} d\Gamma \tag{13}$$

In order to obtain the discrete equations from the weak form, the trial function **u** and the test function δ**u** are approximated by MLS schemes in the form (5). The final discrete equations can be obtained by substituting the trial functions and test functions into the weak form (13), yielding the following system of linear algebraic equations:

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{q} \end{Bmatrix} \tag{14}$$

where

$$\mathbf{K}_{ij} = \int_\Omega \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega, \ \mathbf{f}_i = \int_{\Gamma_t} \phi_i \overline{\mathbf{t}} d\Gamma + \int_\Omega \phi_i \mathbf{b} d\Omega \tag{15}$$

and, in addition,

$$\mathbf{G}_{ik} = -\int_\Omega \phi_i \mathbf{N}_k d\Gamma, \ \mathbf{q}_k = -\int_\Omega \mathbf{N}_k \overline{\mathbf{u}} d\Gamma \tag{16}$$

3. Results and Discussion

The next case analysed is that of a 3D beam loaded uniformly on two opposite sides, as illustrated in Fig. 1. A uniform traction *t* of magnitude *t*= 1.0 per unit area was applied at the lateral sides of the element. The problem was solved considering the symmetry, namely, the appropriate symmetry boundary conditions were applied to the two symmetry planes. The material properties of the beam are chosen as follows: Young's Modulus = 100 and Poisson's ratio =

0.25 and 0.4999. The geometric data was assumed as: length, L = 12a, height, H= 3a, and width, W = 3a, where a = 10.

The problem was solved using EFG formulation and Lagrange multipliers for imposing of the boundary conditions. MLS interpolation is used to interpolate the field variable; MLS functions are constructed using cubic spline weight functions with circular supports and linear polynomial approximation. The deformed quarter beam using Poisson's ratio = 0.25 is plotted in Fig. 2 (the mesh is only for plot purposes).

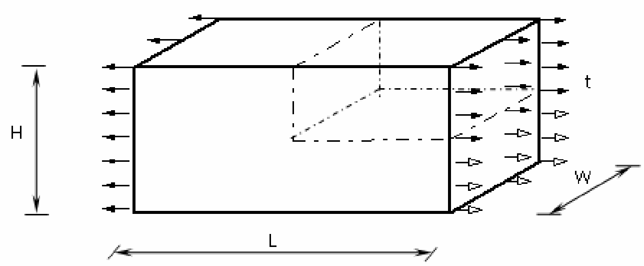

Figure 1: Uniform axial traction element: geometry and load

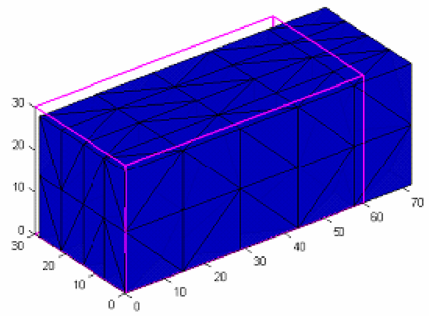

Figure 2: Deformed element subject to uniform axial loading

Fig. 3a shows the convergence in strain energy when a h-type refinement is performed using the EFG. The strain energy of the limit solution of the system is obtained by solving the same problem using a 50 x 50 mesh of 9-noded finite elements.

Fig. 3b shows the convergence curves for the relative errors of the displacement at the middle right face when the same problem is solved using 9-noded finite elements.

Two important observations can be made from Fig. 3; First, for a Poisson's ratio of 0.25, the EFG exhibits a much higher rate of convergence than the one obtained in the classical finite element analysis when unstructured meshes are used. This is due to the fact that continuous approximations are used in the EFG but it may also result from the robustness of the method and the accuracy of the integration.

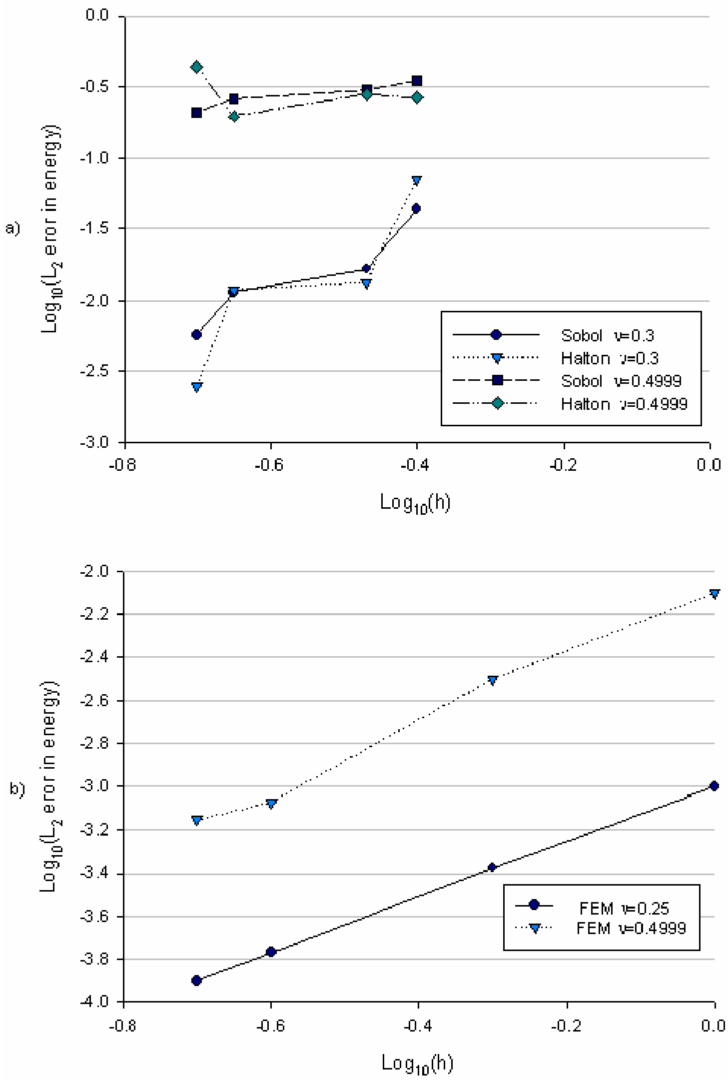

Figure 3: Convergence in strain energy obtained using Sobol and Halton quasi-random integration sequences. a) EFG method and b) Finite element method (using 9-noded displacement based finite element)

The second observation is that when the Poisson's ratio is increased to 0.4999, the finite element solution as well as the solution obtained from the EFG "locks", as evidenced by deterioration in the rate of convergence. This implies that the displacement-based EFG suffers from exactly the same problem as the classical finite element techniques. The reason why a purely displacement-based formulation "locks" is the following: in a pure displacement-based formulation the computed displacement field needs to satisfy the constraint of very small volumetric strains (which become zero as the condition of total incompressibility is approached) while the pressure is of the order of the boundary tractions. The displacement approximation space is not rich enough to accommodate this constraint without a drastic reduction in the rate of convergence.

4. Conclusions

The construction of meshless approximations was reviewed and the effect of numerical integration errors on the solutions was presented. For arbitrary grids the meshless shape functions are rational functions with compact support in the domain. Hence, they are not integrated as accurately by Gauss quadrature. To overcome these difficulties, Quasi-Monte Carlo integration methods is proposed to perform the quadrature. The method is applicable to any type of meshless methods with any number of dimensions and has the advantage of not increasing the complexity even for the 3-D case. Here, we have presented a numerical test for a 3-D elasticity problem. Then a comparison with the classical FEM will be made of the effectiveness of the integration technique. A fully optimized code based on such a computational procedure would surely have a long life since it could easily evolve as meshless technology progresses.

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METODE DE DISCRETIZARE FĂRĂ RETEA ÎN FORMULAREA GALERKIN PENTRU REZOLVAREA GRINZILOR 3-D

(Rezumat)

Principiul metodei de discretizare fără reţea (meshless) este de a aproxima câmpul deplasărilor printr-o combinaţie liniară de funcţii de aproximare construite ţinând cont doar de localizarea intr-un punct în spaţiu, şi nu de o reţea de elemente. Prin urmare, domeniul de calcul va fi discretizat doar prin noduri, fără a fi necesară reţeaua de elemente finite. Funcţiile de aproximare asociate fiecărui nod sunt definite cu ajutorul funcţiilor de pondere ale căror domenii de influenţă se intersectează, nemaifiind astfel necesară asigurarea condiţiilor de continuitate a deplasărilor intre între elemente, ca in metoda clasică de discretizare prin elemente finite.
Tehnica de integrare Monte-Carlo pentru integrarea coeficienţilor matricei sistemului de ecuaţii este promiţătoare in contextul metodelor de discretizare fără reţea. Se studiază implementarea în metoda fără reţea EFG a tehnicii de integrare Monte-Carlo, eliminându-se necesitatea creării unei reţele pentru integrarea termenilor sistemului de ecuaţii. Se prezintă un exemplu numeric pentru calculul grinzilor 3-D bazate pe teoria elasticităţii. Se studiază acurateţea şi convergenţa tehnicii de integrare Monte-Carlo implementată.