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BEAM ELEMENTS ON LINEAR VARIABLE TWO-PARAMETER ELASTIC FOUNDATION

BY

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Abstract. The traditional way to overcome the shortcomings of the Winkler foundation model is to incorporate spring coupling by assemblages of mechanical elements such as springs, flexural elements (beams in one-dimension, 1-D, plates in 2-D), shear-only layers and deformed, pretensioned membranes. This is the class of two-parameter foundations – named like this because they have the second parameter which introduces interactions between adjacent springs, in addition to the first parameter from the ordinary Winkler's model. This class of models includes Wieghardt, Filonenko-Borodich, Hetényi and Pasternak foundations. Mathematically, the equations to describe the reaction of the two-parameter foundations are equilibrium, and the only difference is the definition of the parameters. In order to analyse the bending behavior of a Euler-Bernoulli beam resting on linear variable two-parameter elastic foundation a (displacement) Finite Element (FE) formulation, based on the cubic displacement function of the governing differential equation, is introduced.
Key words: beams, elastic foundations, finite element method.

1. Introduction

The concept of beams and slabs on elastic foundations has been extensively used by geotechnical, pavement and railroad engineers for foundation design and analysis. The analysis of structures resting on elastic foundations is usually based on a relatively simple model of the foundation's response to applied loads.

Generally, the analysis of bending of beams on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point. The vertical deformation characteristics of the foundation are defined by means of continuous, closely spaced linear springs. The constant of proportionality of these springs is known as the modulus of subgrade reaction, k_0. This simple representation of elastic foundation was introduced by Winkler in 1867. The Winkler model (one parameter model), which has been originally developed for

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the analysis of railroad tracks, is very simple but does not accurately represents the characteristics of many practical foundations. One of the most important deficiencies of the Winkler model is that a displacement discontinuity appears between the loaded and the unloaded part of the foundation surface. In reality, the soil surface does not show any discontinuity (Fig. 1).

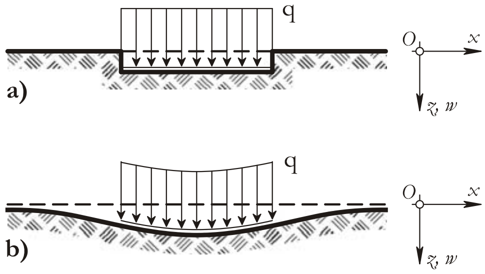

Fig. 1 – Deflections of elastic foundations under uniform pressure:
a – Winkler foundation; b – practical soil foundations.

In order to eliminate the deficiency of Winkler model, improved theories have been introduced on refinement of Winkler's model, by visualising various types of interconnections such as shear layers and beams along the Winkler springs [4] (Filonenko-Borodich (1940); Hetényi (1946); Pasternak (1954); Kerr (1964)). These theories have been attempted to find an applicable and simple model of representation of foundation medium.

Two-parameter foundation models are more accurate than the one-parameter (e.g. Winkler) foundation model. As a special case if the second parameter is neglected, the mechanical modeling of the foundation converges to the Winkler formulation.

2. Basic Assumptions and Analytical Formulation

In what follows we consider straight beams with constant section loaded by forces placed in a principal plane of inertia and continuously supported on a deformable elastic foundation. Beam material is linearly elastic, homogeneous, isotropic and continuous. The foundation medium is assumed to be linear, homogeneous, and isotropic.

The considered beam, supported by a two-parameter elastic foundation, is represented in Fig. 2. The reactive pressure of the two-parameter foundation subjected to a distributed load, *q(x)* is described by [4]

$$p(x) = k_0 B w(x) - k_1 B \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = k w(x) - \bar{k}_1 \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \tag{1}$$

where: B is the width of the beam cross section; w – deflection of the centroidal line of the beam.

For the case of a (linear) variable subgrade coefficients, eq. (1) may be written as

$$p(x) = k_0(x)Bw(x) - k_1(x)B\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = \\ = k(x)w(x) - \bar{k}_1(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \tag{2}$$

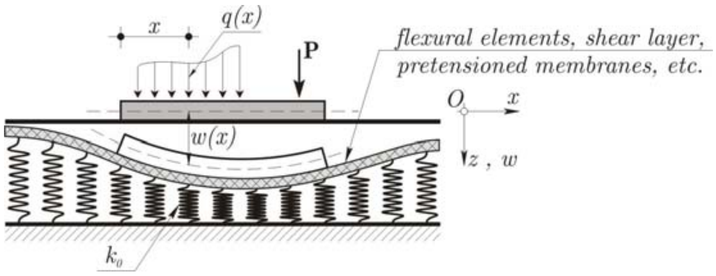

Fig. 2 – Beam resting on two-parameter elastic foundation

The governing equations of the centroidal line of the deformed beam resting on elastic foundation is [7]

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x) - p(x), \tag{3}$$

or substituting *p(x)* from (2),

$$EI\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} + k(x)w(x) - \bar{k}_1(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = q(x), \tag{4}$$

where: E is the modulus of elasticity for the constitutive material of the beam; I – the moment of inertia for the cross section of the beam.

3. FE Formulation

The assumptions and restrictions underlying the development are the same as those of elementary beam theory with the addition of

1. The element is of length l and has two nodes, one at each end.
2. The element is connected to other elements only at the nodes.
3. Element loading occurs only at the nodes.

The beam is divided into m unidimensional finite elements and to each i node of their interconnection, two degrees of freedom are allowed: D_{iw} – the vertical displacement and $D_{i\theta}$ – the slope of cross section. The $\{D\}$ vector of positive nodal displacements is build just like in the system of xOz general axes from Fig. 3.

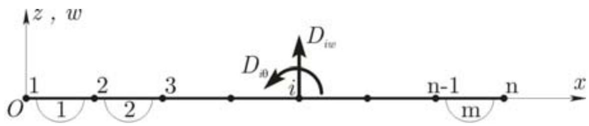

Fig. 3 – FE discretization of the beam domain

In the same way the vector of external nodal actions is build namely

(5) $$\begin{aligned}\{D\} &= \{D_{1w}\, D_{1\theta}\ \ldots\ D_{iw}\, D_{i\theta}\ \ldots\ D_{nw} D_{n\theta}\}^T \\ \{P\} &= \{P_{1w} P_{1\theta} \ldots P_{iw} P_{i\theta} \ldots P_{nw} P_{n\theta}\}^T\end{aligned},$$

To each one dimensional element of beam type, two degrees of fredom are allowed at both extremities: deflection, w_1 and slope, θ_1, and w_2 , θ_2 respectively, positives in the system of local axes from Fig. 4.

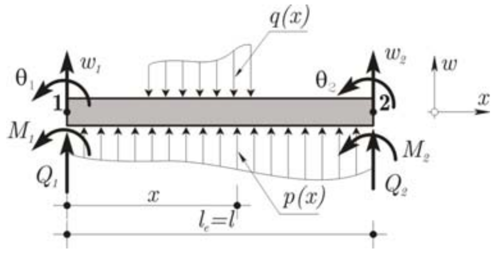

Fig.4 – The FE study

By the help of these displacements, the $\{d_e\}$ vector of elemental nodal displacements and, similarly, the $\{S_e\}$vector of elemental nodal forces, with respect to the system of local axes, are defined

(6) $$\{d_e\} = \{w_1\ \theta_1\ w_2\ \theta_2\}^T\ ,\ \{S_e\} = \{Q_1\ M_1\ Q_2\ M_2\}^T\ ,$$

We must note that Q_1 and Q_2 from (6) are not simply the transverse shear forces in the beam; they includes also the shear resistance associated with modulus of the two-parameter foundation [8]. Force Q_i (i = 1, 2), is a generalized shear force defined by

(7) $$Q_i = V_i + V_i^*\ ,$$

were: $V_i = EI\dfrac{\mathrm{d}^3 w(x)}{\mathrm{d}x^3}$ is the usual shear contribution from elementary beam theory; $V_i^* = -\bar{k}_1 \dfrac{\mathrm{d}w(x)}{\mathrm{d}x}$ – the shear contribution from two-parameter elastic foundation (negative sign arises because a positive slope requires opposite shear forces in the foundation) [8].

Considering the four boundary conditions and the one-dimensional nature of the problem in terms of the independent variable, we assume the displacement function in the form:

(8) $$w_e(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

For the both foundation parameters a linear variation is considered

(9) $$k(x) = k_{(1)} + \frac{k_{(2)} - k_{(1)}}{l}x, \quad \bar{k}_1(x) = \bar{k}_{1,(1)} + \frac{\bar{k}_{1,(2)} - \bar{k}_{1,(1)}}{l}x$$

The choice of a cubic function to describe the displacement is not arbitrary. With the specification of four boundary conditions, we can determine no more than four constants in the assumed displacement function. The second derivative of the assumed displacement function, $w_e(x)$ is linear; hence, the bending moment varies linearly, at most, along the length of the element. This is in accord with the assumption that loads are applied only at the element nodes.

Applying the boundary conditions

(10) $$\begin{cases} w_e(x)\big|_{x=x_1} = w_1, \quad w_e(x)\big|_{x=x_2} = w_2, \\ \left.\dfrac{\mathrm{d}w_e}{\mathrm{d}x}\right|_{x=x_1} = \theta_1, \quad \left.\dfrac{\mathrm{d}w_e}{\mathrm{d}x}\right|_{x=x_2} = \theta_2, \end{cases}$$

in succession, yields:

(11) $$\begin{cases} w_e(x)\big|_{x=0} = w_1 = a_0, \quad w_e(x)\big|_{x=l} = w_2 = a_0 + a_1 l + a_2 l^2 + a_3 l^3, \\ \left.\dfrac{\mathrm{d}w_e}{\mathrm{d}x}\right|_{x=0} = \theta_1 = a_1, \quad \left.\dfrac{\mathrm{d}w_e}{\mathrm{d}x}\right|_{x=l} = \theta_2 = a_1 l + 2a_2 l + 3a_3 l^2. \end{cases}$$

Solving the simultaneous equations (11) will give the coefficients of displacement function in terms of the nodal variables, which are substitute in (8) to obtain the expression of the deflection

(12) $$w_e(x) = N_1(x)w_1 + N_2(x)\theta_1 + N_3(x)w_2 + N_4(x)\theta_2 = [N_i]^T\{d_e\},$$

where $N_i(x)$, (i = 1, …, 4) are the interpolation functions (of Hermite type) that describe the distribution of displacement in terms of nodal values in the nodal displacement vector $\{d_e\}$

$$\begin{cases} N_1(x) = 1 - 3\dfrac{x^2}{l^2} + 2\dfrac{x^3}{l^3},\ N_2(x) = x - 2\dfrac{x^2}{l} + \dfrac{x^3}{l^2} \\ N_3(x) = 3\dfrac{x^2}{l^2} - 2\dfrac{x^3}{l^3},\ N_4(x) = -\dfrac{x^2}{l} + \dfrac{x^3}{l^2} \end{cases} \tag{13}$$

As the polynomial (12) represents an approximate solution of the governing equations (4), it result the residuum (error or discrepancy):

$$\varepsilon(x) = EI\frac{\mathrm{d}^4 w_e(x)}{d\,x^4} - \bar{k}_1(x)\frac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2} + k(x)w_e(x) - q(x) \neq 0 \tag{14}$$

The minimizing of this residuum means to the annulment of Galerkin balanced functional where the weight is considered for each of the four functions, $N_i(x)$

$$\begin{aligned} \Pi_i^e &= \int_0^l N_i(x)\varepsilon(x)dx = \int_0^l N_i(x)\begin{bmatrix} EI\dfrac{\mathrm{d}^4 w_e(x)}{\mathrm{d}x^4} - \bar{k}_1(x)\dfrac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2} + \\ +k(x)w_e(x) - q(x) \end{bmatrix}\mathrm{d}x = \\ &= EI\int_0^l N_i(x)\frac{\mathrm{d}^4 w_e(x)}{\mathrm{d}x^4}\mathrm{d}x - \int_0^l N_i(x)\bar{k}_1(x)\frac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2}\mathrm{d}x + \\ &+\int_0^l N_i(x)k(x)w_e(x)\mathrm{d}x - \int_0^l N_i(x)q(x)\mathrm{d}x = 0,\ (i = 1,\ldots,4) \end{aligned} \tag{15}$$

In first integral from (15), utilizing the parts procedure twice and taking into account the differential relations (in FEM sign convention) from elementary beam theory

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M(x)}{EI},\quad \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = \frac{Q(x)}{EI} \tag{16}$$

we obtain

$$\begin{aligned} \Pi_i^e &= N_i(x)Q(x)\Big|_0^l + \frac{\mathrm{d}N_i(x)}{\mathrm{d}x}M(x)\Bigg|_0^l + EI\int_0^l \frac{\mathrm{d}^2 N_i(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2}\mathrm{d}x - \\ &-\int_0^l N_i(x)\bar{k}_1(x)\frac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2}dx + \int_0^l N_i(x)k(x)w_e(x)\mathrm{d}x - \\ &-\int_0^l N_i(x)q(x)\mathrm{d}x = 0,\ (i = 1,\ldots,4) \end{aligned} \tag{17}$$

In second integral from (17), utilizing the parts procedure and taking into account the relations (7) we obtain

$$\Pi_i^e = N_i(x)V(x)\Big|_0^l + \frac{\mathrm{d}N_i(x)}{\mathrm{d}x}M(x)\Bigg|_0^l + EI\int_0^l \frac{\mathrm{d}^2N_i(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2w_e(x)}{\mathrm{d}x^2}\mathrm{d}x +$$

$$+\int_0^l N_i(x)\frac{\mathrm{d}\bar{k}_1(x)}{\mathrm{d}x}\frac{\mathrm{d}w_e(x)}{\mathrm{d}x}dx + \int_0^l \frac{\mathrm{d}N_i(x)}{\mathrm{d}x}\bar{k}_1(x)\frac{\mathrm{d}w_e(x)}{\mathrm{d}x}\mathrm{d}x + \tag{18}$$

$$+\int_0^l N_i(x)k(x)w_e(x)\mathrm{d}x - \int_0^l N_i(x)q(x)\mathrm{d}x = 0,\ (i = 1,\ldots,4)$$

or in matrix notation,

$$(\ [k_e]+[k_W]+[k_T]\ )\{d_e\} = \{S_e\} - \{R_e\} \tag{19}$$

The last relation represents the elemental physical relation of the one-dimensional finite element of beam on two-parameter elastic foundations, where: $[k_e]$ is the stiffness matrix of the flexure beam element; $[k_W]$ the stiffness matrix of springs layer; $[k_T]$ – the stiffness matrix of the second subgrade parameter; $\{R_e\}$ – the reactions vector of double embeded beam from distributed loads on the element.

The terms of $[k_e]$ matrix are calculated using the relation [1], [2], [3]

$$k_{e,ij} = EI\int_0^l \frac{\mathrm{d}^2N_i(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2N_j(x)}{\mathrm{d}x^2}\mathrm{d}x = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{20}$$

The terms of $[k_W]$ matrix are calculated using the generic relation [3]

$$\begin{aligned} k_{W,ij} &= \int_0^l k(x)N_i(x)N_j(x)\mathrm{d}x = \\ &= \int_0^l \left(k_{(1)} + \frac{k_{(2)} - k_{(1)}}{l}x\right)N_i(x)N_j(x)\mathrm{d}x,\ (i,j = 1,\ldots,4) \end{aligned} \tag{21}$$

resulting (with notations $k_{(1)} = f_1$ and $k_{(2)} = f_2$)

$$[k_W] = \tag{22}$$
$$= \frac{l}{840}\begin{bmatrix} 24(10f_1+3f_2) & 2l(15f_1+7f_2) & 54(f_1+f_2) & -2l(7f_1+6f_2) \\ 2l(15f_1+7f_2) & l^2(5f_1+3f_2) & 2l(6f_1+7f_2) & -3l^2(f_1+f_2) \\ 54(f_1+f_2) & 2l(6f_1+7f_2) & 24(3f_1+10f_2) & -2l(7f_1+15f_2) \\ -2l(7f_1+6f_2) & -3l^2(f_1+f_2) & -2l(7f_1+15f_2) & l^2(3f_1+5f_2) \end{bmatrix}$$

In particular, if in (22) it is assumed that $f_1 = f_2 = k$, one can abtain the stiffness matrix corresponding to the case when first subgrade parameter is constant under the element:

$$[k_W] = \frac{kl}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{23}$$

The terms of $[k_T]$ matrix are calculated using the generic relation:

$$\begin{aligned} k_{T,ij} &= \int_0^l \frac{\mathrm{d}\bar{k}_1(x)}{\mathrm{d}x} N_i(x) \frac{\mathrm{d}N_j(x)}{\mathrm{d}x}\mathrm{d}x + \int_0^l \bar{k}_1(x)\frac{\mathrm{d}N_i(x)}{\mathrm{d}x}\frac{\mathrm{d}N_j(x)}{\mathrm{d}x}\mathrm{d}x = \\ &= \frac{\mathrm{d}\left(\bar{k}_{1,(1)} + \dfrac{\bar{k}_{1,(2)} - \bar{k}_{1,(1)}}{l}x\right)}{\mathrm{d}x} N_i(x)\frac{\mathrm{d}N_j(x)}{\mathrm{d}x}\mathrm{d}x + \\ &+ \left(\bar{k}_{1,(1)} + \frac{\bar{k}_{1,(2)} - \bar{k}_{1,(1)}}{l}x\right)\frac{\mathrm{d}N_i(x)}{\mathrm{d}x}\frac{\mathrm{d}N_j(x)}{\mathrm{d}x}\mathrm{d}x,\ (i,j = 1,\ldots,4) \end{aligned} \tag{24}$$

resulting (with notations $\bar{k}_{1,(1)} = t_1$ and $\bar{k}_{1,(2)} = t_2$) [6]:

$$[k_T] = \frac{1}{30l}\begin{bmatrix} 3(11t_1 + t_2) & -3l(t_1 - 2t_2) & -3(11t_1 + t_2) & 3l(2t_1 - t_2) \\ 3lt_1 & l^2(3t_1 + t_2) & -3lt_1 & -l^2 t_2 \\ -3(t_1 + 11t_2) & 3l(t_1 - 2t_2) & 3(t_1 + 11t_2) & -3l(2t_1 - t_2) \\ 3lt_2 & -l^2 t_1 & -3lt_2 & l^2(t_1 + 3t_2) \end{bmatrix} \tag{25}$$

In particular, if in (22) it is assumed that $t_1 = t_2 = \bar{k}_1$, one can obtain the stiffness matrix corresponding to the case when second subgrade parameter is constant under the element [8]

$$[k_T] = \frac{\bar{k}_1}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \tag{26}$$

The vector $\{R_e\}$ depends on the distributed load on the element and, for $q(x) = q = const.$, it result

$$\{R_e\} = \int_0^l N_i(x) q(x)\mathrm{d}x = \left\{\frac{ql}{2}\ \frac{ql^2}{12}\ \frac{ql}{2}\ -\frac{ql^2}{12}\right\}^T. \tag{27}$$

4. Conclusions

The paper presents a two-node beam element which is obtained by using a cubic Hermitian polynomial to interpolate nodal values of the displacements field and can account in a consistent form for the bearing soil inhomogeneity by considering a linear variation of both foundation parameter. The stiffness matrix and load vector are obtained by using Galerkin's Residual Method and adding the contribution of the foundation as element foundation stiffness matrices to the regular flexure beam element. The obained stiffness matrices are easy to use in modern computer codes.

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ELEMENT FINIT TIP GRINDĂ REZEMATĂ PE MEDIU ELASTIC DESCRIS PRIN DOI PARAMETRI LINIARI VARIABILI

(Rezumat)

Procedeul consacrat în dezvoltarea modelelor folosite pentru studiul fenomenului de interacţiune teren–structură constă în asamblarea unor elemente mecanice simple (resorturi, membrane tensionate, straturi de forfecare etc.) cu scopul surprinderii într-o

manieră cât mai reală a comportării masivelor de pământ sub încărcări. Rezultatele unor astfel de asocieri sunt cunoscute în literatură sub denumirea de modele mecanice cu doi parametri. Din această grupă de modele ale terenului de fundare fac parte: modelul Wieghardt, modelul Filonenko-Borodich, şi modelul Pasternak. Din punct de vedere matematic ecuațiile care descriu aceste modele sunt ecuații de echilibru şi singura diferență constă în definirea parametrilor caracteristici. Utilizând tehnica elementelor finite în formulare reziduală, în lucrare se prezintă o modalitate de stabilire a matricelor de rigiditate corespunzătoare elementului finit tip grindă rezemată pe mediu elastic descris prin doi parametri liniari variabili.