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MODEL FOR FLEXIBLE PLATES SUPPORTED ON PILES

BY

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Abstract. The problem of plates supported on an elastic medium, characterized by several stiffness coefficients corresponding to both linear displacements and rotations, is analysed. This model can be adopted for flexible plates supported on piles. Some particular cases, based on the results presented in [1], [2], are discussed and developed. A case study is performed and the main conclusions from design point of view are drawn.

Key Words: Elastic Support, Stiffness Coefficients, Piles.

1. Introduction

The plates continuously supported on an elastic, deformable medium are well enough studied for the case of Winkler’s medium type, characterized by a stiffness coefficient that is in fact the proportionality factor between the reactive force at a point of the supporting medium and the linear displacement, normal to the surface, at the considered point.

In order to satisfy the requirements imposed by some actual and more complex circumstances, the medium model with several stiffness coefficients corresponding to both linear displacements and rotations, have been introduced. Some studies concerning the design of plates based on this model have been performed.

Further on, this model of the elastic medium, characterized by several stiffness coefficients, one of them corresponding to Winkler’s model and the others being the proportionality factors between the continuously distributed torques and horizontal pressure, respectively that occur on the surface and the corresponding slopes of the deformed contact surface, are more profoundly studied.

A practical case that can be analysed by using this model is the case of plates of mat type, supported on piles [1], [2], [4]. For the plates that have the ratio between the minimum dimension in their own plane and the thickness greater than five, the mat plate must be considered as flexible.

Some constructions provided with such a foundation type have walls of great stiffness, as buildings with structural walls or rigid basements made of concrete, tanks, silos, etc. In these cases the mat can be considered as simply supported or built in the walls, having also vertical displacements, which must be taken into account.

2. Relations between the Internal Forces at the Pile Head and the Corresponding Displacements

The long piles, frequently exceeding 10 m and generally made of pre-cast reinforced concrete are introduced in the ground by driving. In the present and frequently used technologies, the piles are built in the mat.

The head of the pile built in the plate has two linear displacements, one in the direction of its longitudinal axis and the other one in the normal direction to its longitudinal axis and an angular displacement (Fig. 1).

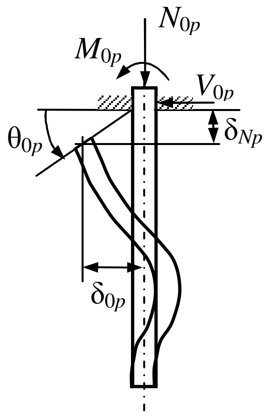

Fig. 1. – Internal forces and displacements.

The model for the pile while it is introduced in the ground is that of an element subjected to an axial force that is resisted by the head of the pile, by the lateral friction that occurs and the surrounding soil deformation.

Between the axial force at the pile head, N_{0p}, and its axial displacement, δ_{Np}, the following relation can be expressed as

$$N_{0p} = k_N \frac{E_p A_p}{l_p} \delta_{Np} = \rho_N \delta_{Np}, \tag{1}$$

where E_p, A_p, l_p are the longitudinal modulus of elasticity, the cross-sectional area and the length of the pile, respectively; k_N is a coefficient experimentally determined on the test piles and takes into account the soil deformation under the pile head, the lateral friction effect, the deformation of the surrounding soil and eventually the effect produced by the fact that the piles work together.

The head of the pile built in the mat will have a linear displacement in the horizontal direction (normal direction to the pile longitudinal axis direction) and an angular displacement (rotation). The rotation of the pile head in one direction can be considered equal to the slope of the plate deformed middle surface in the same direction (Kirchhoff's model).

When the piles have circular or square cross-section, in a plane parallel to the reference system plane, the internal forces at the pile head are

$$V_{0p} = \rho_{V\delta}\delta_{0p} + \rho_{V\theta}\theta_{0p}, \quad M_{0p} = \rho_{M\delta}\delta_{0p} + \rho_{M\theta}\theta_{0p}, \tag{2}$$

where V_{0p} is the shear force at the pile head, M_{0p} – the bending moment at the pile head, $\rho_{V\delta}$, $\rho_{V\theta}$, $\rho_{M\delta}$, $\rho_{M\theta}$ – the pile stiffnesses with respect to the axis the bending occurs. These stiffnesses also depend on the lateral rigidity of the soil. The first subscript indicates the stiffness type: V – shear force, M – bending moment, while the second subscript shows the unit displacement type: δ – deflection ($\delta = 1$) and θ – rotation ($\theta = 1$).

When the soil lateral rigidity coefficient is constant in depth and the pile is considered a semi-infinite bar introduced in an elastic medium, the following relations are obtained for these stiffnesses:

$$\rho_{V\delta} = \frac{k'b_p}{\alpha_p}, \quad \rho_{V\theta} = \rho_{M\delta} = \frac{k'b_p}{2\alpha_p^2}, \quad \rho_{M\theta} = \frac{k'b_p}{2\alpha_p^3}, \tag{3}$$

where the terms have the following significances: k' – the coefficient of soil lateral rigidity, b_p – the pile dimension in the direction normal to the bending plane, α_p – damping coefficient evaluated as follows

$$\alpha_p = \left(\frac{k'b_p}{4E_pI_p}\right)^{\frac{1}{4}}, \tag{4}$$

where E_p represents the longitudinal modulus of elasticity for the pile and I_p – the moment of inertia of the pile cross-section with respect to its own principal centroidal axis, normal to the bending plane.

3. Supporting Medium Description

It is presumed that the piles are identical, having the same stiffnesses in axial and transverse directions. They are uniformly distributed and considered as built in plate.

The discrete connection between plate and piles is substituted by an equivalent continuous medium, that is, the stiffnesses of the piles in axial (vertical) and transverse directions are uniformly distributed over the corresponding contact surface, Ω.

In the vertical direction it is count only on the strength of piles and the pressure on the medium is expressed as

$$p_n = \frac{N}{\Omega} = \frac{\rho_N}{\Omega}\delta_{Np} = k_N\delta_{Np}. \tag{5}$$

The horizontal pressure acting on the medium and the intensity of the distributed torques there are obtained from the ratios

$$p_t = \frac{V_{Op}}{\Omega}, \quad m^* = \frac{M_{0p}}{\Omega}, \tag{6}$$

or, by taking into account relations (2),

$$\begin{cases} p_t = \dfrac{\rho_{V\delta}}{\Omega}\delta_{0p} + \dfrac{\rho_{V\theta}}{\Omega}\theta_{0p} = k_{V\delta}\delta_{0p} + k_{V\theta}\theta_{0p}, \\ m^* = \dfrac{\rho_{M\delta}}{\Omega}\delta_{0p} + \dfrac{\rho_{M\theta}}{\Omega}\theta_{0p} = k_{M\delta}\delta_{0p} + k_{M\theta}\theta_{0p}. \end{cases} \tag{7}$$

In case of actions with normal direction to the middle plane of the plate supported on piles, when the assumption of the normal line element (Kirchhoff) is admitted, it results that the displacement, δ_{0p}, is produced by the pile rotation, θ, in the considered direction, so that

$$\delta_{0p} = \frac{h}{2}\theta. \tag{8}$$

The pile being built in the plate $\theta_{0p} = \theta$,

$$\delta_{0p} = \frac{h}{2}\theta_{0p}. \tag{9}$$

By substituting relations (8) in (6) the following expressions are obtained:

$$\begin{cases} p_t = \left(k_{V\delta}\dfrac{h}{2} + k_{V\theta}\right)\theta_{0p} = k_{t\theta}\theta_{0p}, \\ m^* = \left(k_{M\delta}\dfrac{h}{2} + k_{M\theta}\right)\theta_{0p} = k_{m\theta}\theta_{0p}. \end{cases} \tag{10}$$

From relations (5) and (9) it results that the hypothetic medium with its imposed constraints is characterized by stiffness coefficients, which permit the evaluation of medium reactions upon the plate. These reactive forces (Fig. 2) may be expressed in terms of plate displacements, w and θ, namely

$$-p_n = kw, \quad -p_t = k_t\theta, \quad -m = k_m\theta, \tag{11}$$

where

$$k = k_N, \quad k_t = k_{t\theta}, \quad k_m = k_{m\theta}. \tag{12}$$

By reducing the pressure p_t with respect to the plate middle surface it results

$$\begin{aligned} -m &= -\left(m^* + p_t\frac{h}{2}\right) = -\left(k_m + k_t\frac{h}{2}\right)\theta = \\ &= -\frac{1}{\Omega}\left[\rho_{M\theta} + (\rho_{V\theta} + \rho_{M\delta})\frac{h}{2} + \rho_{V\delta}\frac{h^2}{4}\right]\theta = k^*\theta. \end{aligned} \tag{13}$$

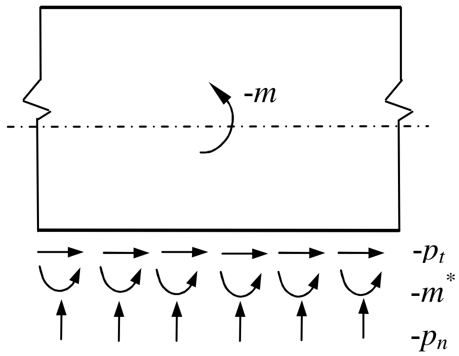

Fig. 2. – Reactive forces.

4. The Differential Equation of Plate Deformed Middle Surface

With respect to a Cartesian coordinate system, for which the xOy plane coincides to the unloaded plate middle surface and z-axis is normal to this plane and has the sense of gravitational acceleration, the deformed middle surface equation has the form $w = w(x,y)$. This is obtained by integrating the differential equation for $k_x^* = k_y^* = k^*$.

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{k^*}{D}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{k}{D}w = \frac{q(x,y)}{D}, \tag{14}$$

or

$$\nabla^2\nabla^2 w + \frac{k^*}{D}\nabla^2 w + \frac{k}{D}w = \frac{q(x,y)}{D}, \tag{15}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{16}$$

$q(x,y)$ represents the intensity of the distributed load acting normal to the plate middle plane.

For same particular support and loading cases equation (14) can be integrated. The relations for internal forces are

$$\begin{cases} M_x = -D\left(\dfrac{\partial^2 w}{\partial x^2} + \nu\dfrac{\partial^2 w}{\partial y^2}\right), \quad M_y = -D\left(\dfrac{\partial^2 w}{\partial y^2} + \nu\dfrac{\partial^2 w}{\partial x^2}\right), \\ M_{xy} = -D(1-\nu)\dfrac{\partial^2 w}{\partial x \partial y}, \quad V_x = -D\left(\dfrac{\partial^3 w}{\partial x^3} + \dfrac{\partial^3 w}{\partial x \partial y^2}\right) + k^*\dfrac{\partial w}{\partial x}, \\ V_y = -D\left(\dfrac{\partial^3 w}{\partial y^3} + \dfrac{\partial^3 w}{\partial x^2 \partial y}\right) + k^*\dfrac{\partial w}{\partial y}. \end{cases} \tag{17}$$

The boundary conditions are valid for simply supported and fixed sides, while for a free side they become:

$$\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial t^2} - \frac{k^*}{D}\frac{\partial w}{\partial n} = 0, \tag{18}$$

where n indicates the normal direction to the contour and t the tangent direction. With a good enough accuracy, there are evaluated

a) the axial force at the pile head

$$N_{0p} \cong \rho_N w; \tag{19}$$

b) the bending moments at the pile head, in x- and y-axes directions

$$M_x = \left(\rho_{M\delta}\frac{h}{2} + \rho_{M\theta}\right)\frac{\partial w}{\partial x}, \quad M_y = \left(\rho_{M\delta}\frac{h}{2} + \rho_{M\theta}\right)\frac{\partial w}{\partial y}. \tag{20}$$

The deflections, w, and the slopes $\partial w/\partial x$ and $\partial w/\partial y$ are computed at the points where the piles are considered to be built in plate.

When the pile stiffness has the same magnitude in any direction, the maximum bending moment at its head is given by the maximum slope. It is firstly determined the angle β that defines the direction along which the slope has a maximum value.

$$\tan\beta = \frac{\partial w/\partial y}{\partial w/\partial x} \tag{21}$$

Further, the maximum slope is evaluated

$$\left(\frac{\partial w}{\partial n}\right)_{\max} = \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}. \tag{22}$$

The maximum bending moment will be

$$M_{\max} = \left(\rho_{M\delta}\frac{h}{2} + \rho_{M\theta}\right)\sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}. \tag{23}$$

The shear force at the pile head has also the maximum value in the direction of the maximum slope and can be determined by using the relation

$$V_{\max} = \left(\rho_{V\delta}\frac{h}{2} + \rho_{V\theta}\right)\sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}. \tag{24}$$

5. Application

It is considered a square plate (12 m × 12 m), 40 cm in thickness, supported on 100 piles. The distance between two consecutive piles in both directions is 120 cm.

The whole system being symmetrical with respect to plate diagonals, the results for only the eight part of the plate are presented.

The contour of the plate is acted by a global force $P = 30,000$ kN that produces an uniform displacement along the plate boundaries, equal to 10 mm. The force is transmitted to the plate through a tubular rigid structure, in vertical direction.

There are determined the plate deflections (Table 1) by using the finite differences method. The mesh of the adopted finite differences grid is $\Delta = 60$ cm (Fig. 3). The deflection graphs along directions *1-1* and *2-2* are represented in Fig. 4.

Table 1
Nodal Deflections

Node	w, [cm]	Node	w, [cm]	Node	w, [cm]	Node	w, [cm]	Node	w, [cm]
1	1.0000	14	0.5604	27	0.0510	40	0.6854	53	0.5172
2	0.7877	15	0.3397	28	−0.0172	41	0.4561	54	0.4423
3	0.5543	16	0.1693	29	−0.0541	42	0.2794	55	0.3185
4	0.3393	17	0.0153	30	−0.0716	43	0.1501	56	0.2285
5	0.1704	18	−0.0193	31	1.0000	44	0.0642	57	1.0000
6	0.0533	19	−0.0591	32	0.7584	45	0.0147	58	0.6303
7	−0.0196	20	−0.0785	33	0.5517	46	1.0000	59	0.4817
8	−0.0602	21	−0.0865	34	0.3052	47	0.6060	60	0.3787
9	−0.0802	22	1.0000	35	0.1529	48	0.4240	61	1.0000
10	−0.0886	23	0.8292	36	0.0516	49	0.2794	62	0.7374
11	−0.0907	24	0.5627	37	−0.0088	50	0.1718	63	0.5803
12	1.0000	25	0.3331	38	−0.0402	51	0.1012	64	1.0000
13	0.8016	26	0.1639	39	1.0000	52	1.0000	65	0.8513

There are also evaluated the axial forces and bending moments in the most loaded columns that, as expected, are the columns 65, 58, 47, 32 and 13. For these columns the axial force, N is around 700 kN, while the bending moment, My, equals 60 kNm.

It can be pointed out that the piles are subjected to combined compression and bending, this last effect being enough important.

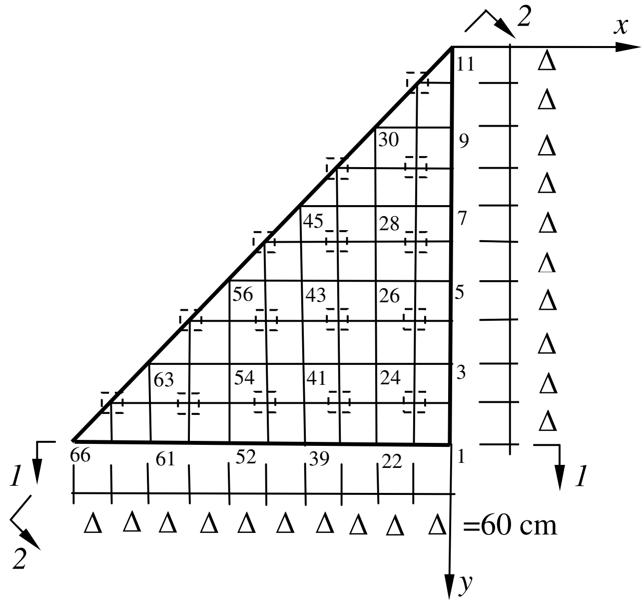

Fig. 3. – Finite differences mesh.

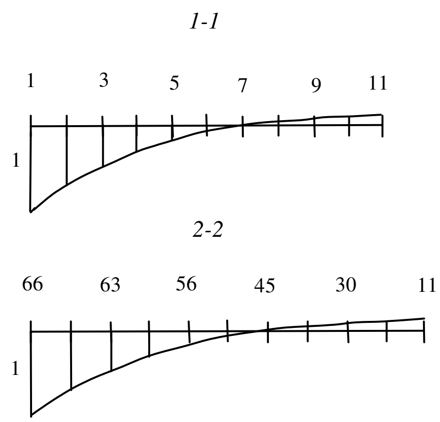

Fig. 4. – Deflection graphs.

6. Conclusions

A plate on dense supports that transmit forces proportional to plate linear displacements and couples proportional to plate rotations, is substituted by an equivalent continuous medium, which represents a generalization of plates supported on elastic medium.

This model can be successfully applied for plates supported on uniformly distributed piles. In this approach, the interaction between the flexible plate and the support system is adequately considered.

The design model permits the evaluation not only of axial forces in the piles but also of bending moments at the head of piles.

The presented numerical example shows that the piles are generally non-uniformly loaded and the moments at the heads of piles have important values.

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MODEL PENTRU PLĂCI FLEXIBILE REZEMATE PE PILOŢI

(Rezumat)

Se abordează problema elementelor bidimensionale rezemate pe mediu elastic ce este caracterizat de coeficienţi de rigiditate corespunzători atât deplasărilor liniare cât şi celor unghiulare. Modelul poate fi adoptat pentru plăcile flexibile (radiere) rezemate pe piloţi, a căror legătură discretă cu radierul este substituită cu o legătură continuă echivalentă. Se face şi un studiu de caz, determinându-se săgeţile unui radier flexibil prin metoda diferenţelor finite.