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DIFFERENTIAL EQUATION OF A VISCO-ELASTIC BEAM SUBJECTED TO BENDING

BY

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Abstract. Generally, building materials exhibit a rheological behaviour as a direct consequence of their fundamental material properties such as elasticity, viscosity and plasticity. Based on the Kelvin–Voight model, governing the behaviour of the visco-elastic materials such as concrete, the present paper proposes the differential equation corresponding to the middle line of a visco-elastic beam subjected to bending. The derived equation is then used in the finite element analysis of the beam, also known as the Galerkin method. Following the solving procedure, a system of first order differential equations (expressed in terms of deformations and deformation velocities) is obtained, written in matrix form. The solution of such a system of equations could be obtained by means of the finite differences method.

Key Words: Kelvin-Voight model; Galerkin method; finite differences method.

1. Introduction

The constitutive law of the mechanical behaviour of construction materials could be expressed by the following rheologic eq.

$$f(\sigma, \dot{\sigma}, \ldots, \varepsilon, \dot{\varepsilon}, \ldots, t, T) = 0, \tag{1}$$

that is a function of stresses, strains, their derivatives with respect to the time, t, time and temperature. Furthermore there are also coefficients in eq. (1), called *phenomenologic parameters* that can be either constant or variable in time. It can, therefore, be said that eq. (1) is a highly empirical equation.

In a three dimensional space having σ, ε, t as coordinates, eq. (1) defines a surface called *characteristic surface*. Based on the variation of each parameter, the following three cases could be distinguished:

a) for $t =$ const., the constitutive law defined by the eq.

$$f_1(\sigma, \varepsilon) = 0. \tag{2}$$

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b) for $\varepsilon =$ constant, the constitutive law defined by the eq.

$$f_2(\sigma, t) = 0. \tag{3}$$

c) for $\sigma =$ const., the constitutive law defined by the eq.

$$f_3(\varepsilon, t) = 0. \tag{4}$$

Generally, equation (1) expresses mathematically the combination of the three fundamental mechanical properties of materials, in different percentages, such as

a) *Linear elasticity* (Hooke behaviour) – the elastic deformation goes back to the initial unloaded state (zero deformation) after the load has been removed. Such behaviour could be graphically represented by means of the spring equivalence. Its characteristic curve is shown in Fig. 1*a*,

b) *Viscosity* – characterizes the property of materials to partly recover from their deformed state but with a complete recovery of the deformation speed. Such behaviour is idealized by a dash-pot or damper being characterized by the curve shown in Fig. 1*b*, where η is the viscosity coefficient,

c) *Plasticity* – the material property that defines a continuous deformation even under constant load. It is also characterized by a residual deformation after the load has been removed. Such behaviour could be graphically represented by the rigid-plastic Saint–Venant model, in the form of a sliding mechanism, shown in Fig. 1*c*.

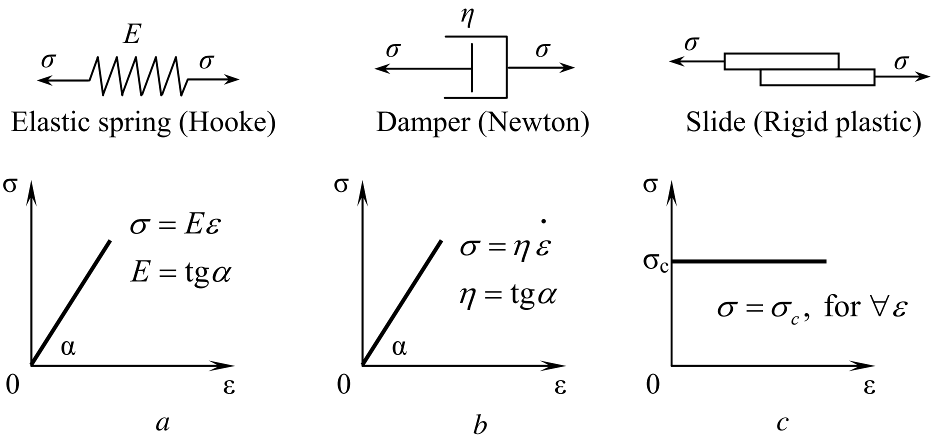


Fig. 1. – Mechanical behaviour models of construction materials.

As it is known, construction materials have a diverse behaviour due to the fact that the three properties presented above are mixed in different percentages. As a consequence, complex behaviour material models have been developed such

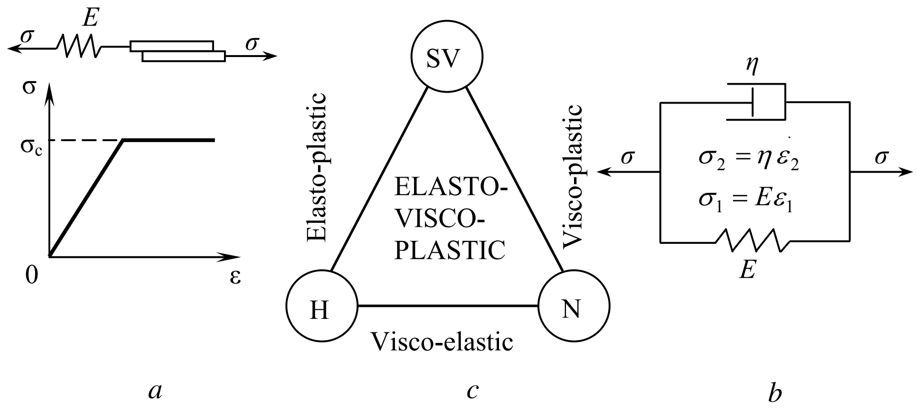


Fig. 2. – Rheological behaviour triangle and complex behaviour material models.

as: Prandtl elasto-plastic model (Fig. 2*a*) and Kelvin–Voigt visco-elastic model (Fig. 2*b*).

The triangle of rheological behaviour gives the possibility to visualize the association and combination of the three fundamental material properties (Fig. 2*c*). The vertexes of an equilateral triangle are represented by each fundamental material property, individually (Hooke, Newton and Saint–Venant). The side joining two vertexes characterizes the materials that have the two properties from the nodes mixed in different percentages. For example, the side joining the vertexes representing the Hooke and Newton models, denoted by H and N, respectively, characterizes the visco-elastic materials. Consequently, the side joining the vertexes representing the Newton and Saint–Venant models, denoted by SV, characterizes the materials with a visco–plastic behaviour. The third side of the triangle, H – SV defines the materials with elasto-plastic behaviour. All the points inside the rim of the triangle represent materials with elasto-visco-plastic behaviour [5].

2. Kelvin–Voigt (K–V) Mechanical Model

The rheologic behaviour of visco-elastic materials can be mathematically expressed by means of differential eqs. having the general form

$$f(\sigma, \dot{\sigma}, \ldots, \varepsilon, \dot{\varepsilon}, \ldots) = 0, \tag{5}$$

in which the above mentioned phenomenological coefficients are taken into account as constants. The values of these coefficients are considered based on equivalent mechanical models that consist in one or more springs and dash-pots connected in series or parallel. The analytical formulation of the behaviour of the system is based on the following two conditions: static equilibrium and compatibility of the deformations. The K–V model, defining the solid bodies, is obtained by linking together, in parallel, a spring (Hooke) and a dash-pot (Newton) as shown in Fig. 2*b*.

The equilibrium of the body can be expressed as

$$\sigma_1 + \sigma_2 = \sigma, \tag{6}$$

and the compatibility equation takes the form

$$\varepsilon_1 = \varepsilon_2 = \varepsilon. \tag{7}$$

The stresses in the spring and in the dash-pot (damper) are written as

$$\sigma_1 = E\varepsilon_1, \sigma_2 = \eta\dot{\varepsilon}_2, \tag{8}$$

and by substituting eqs. (7) and (8) in the expression (6) it follows

$$E\varepsilon_1 + \eta\dot{\varepsilon}_2 = E\varepsilon + \eta\dot{\varepsilon} = \sigma. \tag{9}$$

Eq. (9) can also be written in differential form

$$\left(E + \eta\frac{\mathrm{d}}{\mathrm{d}t}\right)\varepsilon = \sigma, \quad Q(\varepsilon) = P(\sigma), \tag{10}$$

where Q and P are the following differential operators:

$$Q = E + \eta\frac{\mathrm{d}}{\mathrm{d}t} = E + \eta s, \quad P = 1. \tag{11}$$

Taking into account the notation $s^n = \mathrm{d}^n/\mathrm{d}t^n$, it follows that the Q and P operators become polynomials in terms of s.

3. Differential Equation of the Beam Made From Visco-Elastic Material

The static equivalence relationship between the efforts and the stresses on a cross-section is

$$M = \int_A y\sigma \mathrm{d}A. \tag{12}$$

According to Bernoulli's postulate, the linear strain of a fibre located at a distance, y, from the neutral axis is $\varepsilon = \chi y$, where $\chi = 1/\rho$ is the curvature of the neutral axis. From the similarity of the "curved" triangles in Fig. 3*a*, it follows:

$$\frac{y}{\rho} = \frac{\varepsilon_x \mathrm{d}x}{\mathrm{d}x} \quad \chi y = \varepsilon_x. \tag{13}$$

The differential operator with respect to time, P, is applied to eq. (12), leading to

$$P(M) = \int_A yP(\sigma)\mathrm{d}A. \tag{14}$$

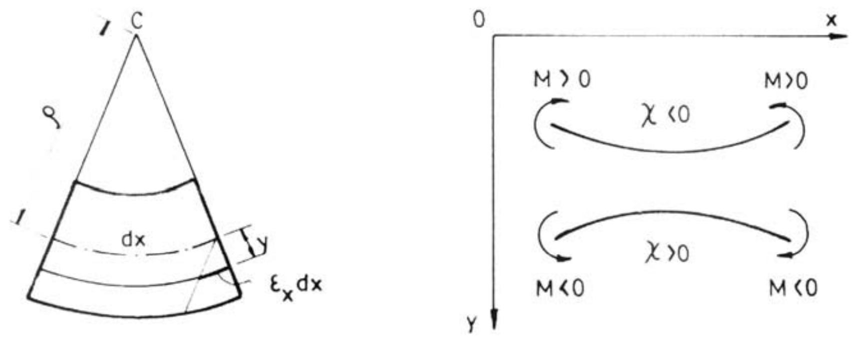
Furthermore, the rheologic eq. (10) can be re-written as

$$P(s)(\sigma) = Q(s)(\varepsilon) = Q(s)(y\chi) = yQ(s)(\chi), \tag{15}$$

and substituting the above expression in eq. (14), it leads to

$$P(M) = \int_A yyQ(s)\chi\mathrm{d}A = I_zQ(s)\chi, \tag{16}$$

where $I_z = \int y^2\mathrm{d}A$ is the moment of inertia (or the second moment of area) with respect to the z-axis of the cross-section.



a *b*

Fig. 3. – Differential element of a beam (*a*) and the sign convention of the curvature (*b*).

For small deformations the substitution $\chi = -v''$ can be made, the sign convention from Fig. 3*b* being taken into account. If χ were substituted in eq. (16) it results

$$P(M) = -I_zQ(s)\chi = -I_zQ(s)v'' \Rightarrow v'' = -\frac{P(s)M}{I_zQ(s)} \tag{17}$$

and taking the second order derivative with respect to x,

$$I_zQ(s)v^{IV} = P(s)p_n, \tag{18}$$

which represents the fourth order differential eq. of the deformed shape of the neutral axis of a beam made of a visco-elastic material. Following the same procedure, the corresponding eq. for an elastic beam is

$$EI_z v^{IV} = P(s)p_n. \tag{19}$$

4. The Finite Element of a Visco-Elastic Beam

A finite element of a K–V material visco-elastic beam is considered (Fig. 4). For each end of the finite element, there are taken into account two degrees of freedom. These degrees of freedom make up the vectors of displacements $\{d_e\} = \left\{v_1 \quad \theta_1 \quad v_2 \quad \theta_2\right\}^T$ and $\{S_e\} = \left\{T_1 \quad M_1 \quad T_2 \quad M_2\right\}^T$ forces of the element. The displacements can be either deflections of rotations and the corresponding forces are the shear forces and the bending moments.

It is assumed that the deflection of the beam changes along the length of the element by a polynomial of the third degree

$$v(x,t) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3, \tag{20}$$

where the coefficients $\alpha_i = \alpha_i(t)$ could also depend on time.

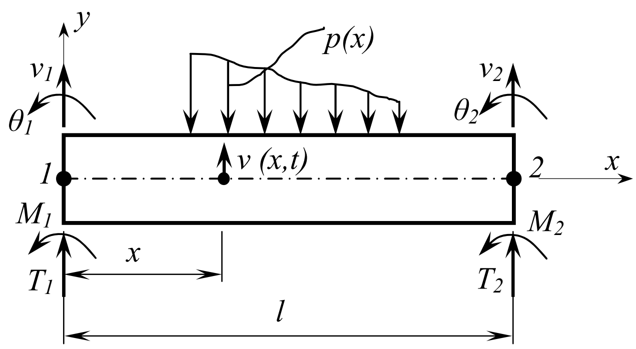


Fig. 4. – The finite element of a visco–elastic beam.

The coefficients $\alpha_i(t)$ are expressed in terms of the nodal displacements $d_e(t)$ from the boundary conditions $v(x=0) = v_1$, $v'(x=0) = \theta_1$, $v(x=l) = v_2$, $v'(x= = l) = \theta_2$

$$v(x,t) = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_1(t) \\ \theta_1(t) \\ v_2(t) \\ \theta_2(t) \end{Bmatrix} = [N_i(x)]^T \{d_e(t)\}, \tag{21}$$

where $N_i(x)$ are the shape functions of Hermite type [1].

The residue can be obtained based on eq. (18) namely

$$\varepsilon(x,t) = IQ(s)v^{IV}(x,t) - P(s)p = 0, \tag{22}$$

and the Galerkin function should be minimized on the finite element

$$\begin{aligned}\Pi_e &= \int_0^l N_i(x)\varepsilon(x,t)\mathrm{d}x = \int_0^l N_i(x)\left[IQ(s)v^{IV}(x,t) - P(s)p\right]\mathrm{d}x = \\ &= IQ(s)\int_0^l N_i(x)v^{IV}(x,t)\mathrm{d}x - P(s)\int_0^l N_i(x)p(x)\mathrm{d}x = 0\end{aligned} \tag{23}$$

For simplicity, starting from eq. (22), the notations $I_z = I$ and $p_n = p$ have been used.

After integrating twice the first member of the sum with respect to x

$$\begin{aligned}I_1 &= IQ(s)N_i(x)v'''(x,t)\big|_0^l - IQ(s)\int_0^l N_i'(x)v'''(x,t)\mathrm{d}x = \\ &= IQ(s)\left[N_i(x)v'''(x,t)\big|_0^l - N_i'(x)v''(x,t)\big|_0^l + \int_0^l N_i''(x)v''(x,t)\mathrm{d}x\right].\end{aligned} \tag{24}$$

The expressions (17) and (18) are taken into account when writing eq. (24) and I_1 is also substituted in the expression of Π_e in order to obtain the relationship

$$\begin{aligned}\Pi_e = &-N_i(x)P(s)T(x,t)\big|_0^l + N_i'(x)P(s)M(x,t)\big|_0^l + \\ &+IQ(s)\left(\int_0^l N_i''(x)N_i''(x)\mathrm{d}x\right)\{d_e(t)\} - P(s)\int_0^l N_i(x)p\mathrm{d}x = 0.\end{aligned} \tag{25}$$

Applying the sign convention from the FEM and after performing all the calculations, a simpler form of the above eq. is obtained

$$P(s)\left(\{S_e\} - \{R_e\}\right) = IQ(s)[k_e]\{d_e\}, \tag{26}$$

where: $[k_e]$ is the stiffness matrix of the finite element with the 4×4 size; $\{R-e\}$ – the vector of the support reactions in a double fixed beam (as a result of the distributed loads over the length of the element). The entries of the stiffness matrix and of the reaction vector are computed using the following relation:

$$k_{ij} = \int_0^l N_i''(x)N_j''(x)\mathrm{d}x, \quad R_i = \int_0^l p(x)N_i(x)\mathrm{d}x,\ (i,j = 1,\ldots,4). \tag{27}$$

The terms k_{ij} and R_i are computed in the same manner as in the elastic case but R_i could still be time dependent provided that $p = p(x,t)$. Equation (26) represents the physical eq. of the finite element of a visco-elastic beam.

In case of the K–V model, $P(s) = 1$ and $Q(s) = E + \eta\,\mathrm{d}/\mathrm{d}t$. Therefore, eq. (26) becomes

$$\left(E + \eta \frac{\mathrm{d}}{\mathrm{d}t}\right) I[k_e]\{d_e\} = \{S_e\} - \{R_e\}, \tag{28}$$

or, after performing further simplifying mathematical operations

$$EI[k_e]\{d_e\} + \eta I[k_e]\{\dot{d}_e\} = \{S_e\} - \{R_e\} = \{F_e\} \tag{29}$$

where $\{F_e\}$ denotes the vector of nodal forces of the finite element.

In order to be able to assembly all the obtained vectors based on eq. (29), for each finite element, to get the vector of nodal displacements of the entire structure, the expansion procedure has to be applied namely

$$EI[k_e^{\exp}]\{D_s\} + \eta I[k_e^{\exp}]\{\dot{D}_s\} = \{F_e^{\exp}\}. \tag{30}$$

The summation of all the terms obtained by means of eq. (29) leads to the constitutive eq. of the entire structure

$$EI[K_s]\{D_s\} + \eta I[K_s]\{\dot{D}_s\} = \{P_s\} - \{R_s\} = \{F_s\}, \tag{31}$$

where: $\{P_s\}$ is the vector of the applied forces at the nodes of the structure and $[K_s]$ and $\{R_s\}$ are computed for the entire structure by means of direct summation

$$[K_s] = \sum_{e=1}^{m} [k_e^{\exp}]; \quad \{R_s\} = \sum_{e=1}^{m} \{R_e^{\exp}\}. \tag{32}$$

From the boundary conditions, eq. (32) becomes

$$EI[K]\{D\} + \eta I[K]\{\dot{D}\} = \{P\} - \{R\} = \{F\}, \tag{33}$$

where: $\{D\}$ is the vector of the free nodal displacements (different from 0) of the entire structure; $\{P\}$ – the vector of the active nodal forces. Equation (33) defines a system of first order differential eqs. with respect to the time, t. Such a system of eqs. can be solved by means of numerical methods such as the finite differences method. For this purpose, the time interval from 0 to t is divided into smaller time steps, Δt. The division points of the time interval are denoted by $i-1$, i, $i+1$ (Fig. 5). The following expression is obtained by using the central differences method:

$$\dot{D}_i = \frac{D_{i+1} - D_{i-1}}{2\Delta t}, \tag{34}$$

and by substituting it in the system of eqs. defined by eq. (33) written in finite differences for the time step $t = i\Delta t$, the following expression is obtained

$$EI[K]\{D_i\} + \eta I[K]\frac{\{D_{i+1}\} - \{D_{i-1}\}}{2\Delta t} = \{F_i\}. \tag{35}$$

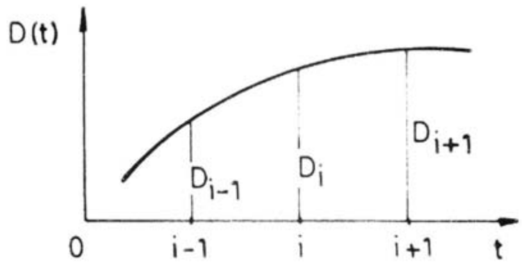


Fig. 5. – Graphical representation of the interval nodes.

After further mathematical calculations the eqs. transforms into

$$\frac{\eta I[K]}{2\Delta t}\{D_{i+1}\} = \{F_i\} + \frac{\eta I[K]}{2\Delta t}\{D_{i-1}\} - EI[K]\{D_i\}, \tag{36}$$

which, in fact, represents a recurrence formula that determines the displacements at the time step $i+1$ if the corresponding displacements at time steps $i-1$ and i were known.

The computational algorithm based on the above mentioned relationship starts for the time step $t = 0$ for which the vector of initial displacements, $\{D_0\}$, is known. Furthermore, the displacement vector *prior* to the initial 0 conditions, $\{D_{-1}\}$, is not known and therefore applying eq. (36) it leads to

$$\frac{\eta I[K]}{2\Delta t}D_1 = \{F_0\} + \frac{\eta I[K]}{2\Delta t}\{D_{-1}\} - EI[K]\{D_0\}. \tag{37}$$

From eq. (34) it follows that

$$\{D_{i+1}\} = \{D_{i-1}\} + 2\Delta t\{\dot{D}_i\}, \tag{38}$$

and when written for the time step $t = 0$, it leads to

$$\{D_1\} = \{D_{-1}\} + 2\Delta t\{\dot{D}_0\} = \{D_1\} - 2\Delta t\{\dot{D}_0\}. \tag{39}$$

Substituting it in the relation (37) and after making the necessary calculations, one reaches the following expression:

$$\{D_0\} = \frac{1}{EI}\left(\{F_0\}[K^{-1}] - \eta I\{\dot{D}_0\}\right), \tag{40}$$

where $\{F_0\}$ and $\{D_0\}$ are known from the initial given conditions. Equation (40) shows that the displacement of the beam at the initial stage can be expressed as functions of the deformation velocity and *vice-versa*. The recurrence formula (36) can be re-written as

$$\{D_{i+1}\} = \{D_{i-1}\} - 2\Delta t \frac{E}{\eta}\{D_i\} + \frac{2\Delta t}{\eta I}[K^{-1}]\{F_i\} \tag{41}$$

in order to allow for the calculation of the displacements for the visco-elastic beam at the time $t = (i+1)\Delta t$ as a function of the displacements from the preceding time steps $i-1$ and i. For the case when the forces $\{F_i\}$ are constant in time, therefore equal to $\{F_0\}$, but the displacements (deflections) of the beam increase with time, it is considered to be the case of slow yielding (also known as creep).

5. Conclusions

The behaviour of materials and structural elements depends on a large number of parameters. In order to simplify the calculus, the equivalent mechanical models take into account only a reduced number of such parameters, namely the ones that are considered to be critical to describing, as accurate as possible, the real behaviour.

The Kelvin-Voigt mechanical model takes into account, besides the property of linear elasticity, the property of material viscosity, which is an important parameter in a heterogeneous material such as concrete.

Equation (18) describing the deformed shape of a beam made of a visco-elastic material is a generalized expression of the relationship characteristic to an elastic beam.

The finite element method (FEM) allows the consideration of the rheological behaviour by means of the shape functions describing the displacement field along the finite element.

The general mathematical model of an entire structure for the visco-elastic Kelvin-Voigt material, derived by means of the FEM, is a system of differential eqs. of the 1st order, coupled with respect to time. Such a system can be solved by various integration methods (in the paper the finite differences method is presents)

The computation algorithm to find the displacements of the visco-elastic beam, at various time steps, can be very efficiently applied in a computer programme.

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ECUAŢIA DIFERENŢIALĂ A UNEI BARE VÂSCOELASTICE SOLICITATE LA ÎNCOVOIERE

(Rezumat)

Materialele de construcţii prezintă, în general, o comportare reologică diversă ca urmare a asocierii, în diverse proporţii, a proprietăţilor fundamentale de elasticitate, vâscozitate şi plasticitate. Pe baza modelului mecanic Kelvin-Voigt, specific materialelor vâscoelastice (printre care se numără şi betonul), în lucrare se stabileşte ecuaţia diferenţială a fibrei medii deformate a unei grinzi vâscoelastice încovoiate, ecuaţie care este utilizată apoi în analiza grinzii prin metoda elementelor finite, procedeul Galerkin. Se ajunge la o ecuaţie matriceală, care reprezintă un sistem de ecuaţii diferenţiale de ordinul 1 (în viteze de deformare şi deplasări), cuplate în raport cu timpul, sistem ce se poate integra, de exemplu, prin metoda diferenţelor finite.