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BEAMS ON ELASTIC FOUNDATION
THE SIMPLIFIED CONTINUUM APPROACH

BY

IANCU-BOGDAN TEODORU

Abstract. The key aspect in the design of flexible structural elements in contact with bearing soils is the way in which soil reaction, referred to qualitatively as soil's reactive pressure (p), is assumed or accounted for in analysis. A magnitude and distribution of p might be preliminary assumed, or some mathematical relationships could be incorporated into the analysis itself, so that p is calculated as part of the analysis. In order to eliminate the bearing soil reaction as a variable in the problem solution, the simplified continuum approach is presented. This idealization provides much more information on the stress and deformation within soil mass compared to ordinary Winkler model, and it has the important advantage of the elimination of the necessity to determine the values of the foundation parameters, arbitrarily, because these values can be computed from the material properties (deformation modulus, E_s, Poisson ratio, v_s and depth of influence zone, H, along the beam) for the soil. A numerical investigation of the simplified-continuum approach is also presented.

Key words: beams; elastic foundations; Winkler foundation; Vlasov foundation; two-parameter elastic foundation; EBBEF2p computer code.

1. Introduction

Generally, the analysis of bending of beams on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point. The vertical deformation characteristics of the foundation are defined by means of identical, independent, closely spaced, discrete and linearly elastic springs. The constant of proportionality of these springs is known as the modulus of subgrade reaction, k_s. This simple and relatively crude mechanical representation of soil foundation was firstly introduced by Winkler, in 1867 [3], [1].

The Winkler model, which has been originally developed for the analysis of railroad tracks, is very simple but does not accurately represents the characteristics of many practical foundations. One of the most important deficiencies of the Winkler model is that a displacement discontinuity appears between the loaded and the unloaded part of the foundation surface. In reality, the soil surface does not show any discontinuity (Fig. 1).

Fig. 1 – Deflections of elastic foundations under uniform pressure: *a* – Winkler foundation; *b* – practical soil foundations.

Historically, the traditional way to overcome the deficiency of Winkler model is by introducing some kind of interaction between the independent springs by visualising various types of interconnections such as flexural elements (beams in one-dimension (1-D), plates in 2-D), shear-only layers and deformed, pretensioned membranes [3]. The foundation model proposed by Filonenko and Borodich in 1940 [3] acquires continuity between the individual spring elements in the Winkler model by connecting them to a thin elastic membrane under a constant tension. In the model proposed by Hetényi in 1950 [3], interaction between the independent spring elements is accomplished by incorporating an elastic plate in 3-D problems, or an elastic beam in 2-D problems, that can deforms only in bending. Another foundation model, proposed by Pasternak in 1954, acquires shear interaction between springs by connecting the ends of the springs to a layer consisting of incompressible vertical elements which deform only by transverse shearing [3]. This class of mathematical models have another constant parameter which characterizes the interaction implied between springs and hence are called *two-parameter models* or, more simply, *mechanical models* (Fig. 2).

Another approach to develop, and also to improve foundation models starts with the three complex sets of differential equations with partial derivatives (compatibility, constitutive, equilibrium) governing the behavior of the soil as a semi-infinite continuum, and then introduce simplifying assumptions with respect to displacements or/and stresses in order to render the remaining equations fairly easy to solve in an exact, closed-form, manner. These are referred to as simplified-continuum models.

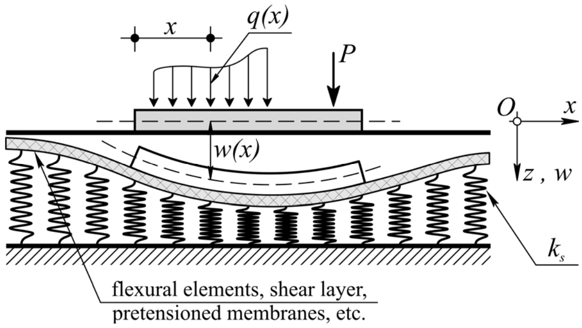


Fig. 2 – Beam resting on two-parameter elastic foundation.

Reissner [4], [5] pioneered a straightforward application of the simplified-continuum concept to produce what is referred to as the Reissner Simplified Continuum model. Assuming a foundation layer in which all in-plane stresses are negligibly small,

$$\sigma_x = \sigma_y = \tau_{xy} = 0\,, \tag{1}$$

and the horizontal displacements at the top and bottom surfaces of the foundation are zero, he obtain the analytical solution for elastic foundation given by

$$q(x,y) - \frac{c_2}{4c_1}\nabla^2 q(x,y) = c_1 w(x,y) - c_2 \nabla^2 w(x,y)\,, \tag{2}$$

where $q(x,y)$ is a distributed load acting on the foundation surface, $w(x,y)$ – the displacement of the foundation surface in the z-direction, and

$$c_1 = \frac{E_s}{H};\; c_2 = \frac{HG_s}{3}\,, \tag{3}$$

were E_s , G_s and H are deformation modulus, shear modulus and depth of the foundation, respectively.

A consequence of assumption (1) is that the shear stresses in the zx- and zy-planes are independent of the z-coordinate and thus are constant through the depth of the foundation, for a given surface point, (x,y). Therefore, this model may be applied only to study the response near loading contact area and not to study stresses inside the foundation [3].

Vlasov, in 1960, adopted the simplified-continuum approach based on the variation principle and derived a two-parameter foundation model [2]. In his method the foundation was treated as an elastic layer and the constraints were imposed by restricting the deflection within the foundation to an appropriate mode shape, $\varphi(z)$. The two-parameters Vlasov model (Fig. 3) accounts for the effect of the neglected shear strain energy in the soil and shear forces that come from surrounding soil by introducing an arbitrary parameter, γ, to characterize the vertical distribution of the deformation in the subsoil [2]; the authors did not provide any mechanism for the calculation of γ. Jones and Xenophontos [2] established a relationship between the parameter γ and the displacement characteristics, but did not suggest any method for the calculation of its actual value. Following Jones and Xenophontos, Vallabhan and Das [7] determined the parameter γ as a function of the characteristic of the beam and the foundation, using an iterative procedure. They named this model a *modified Vlasov model* [7], [8].

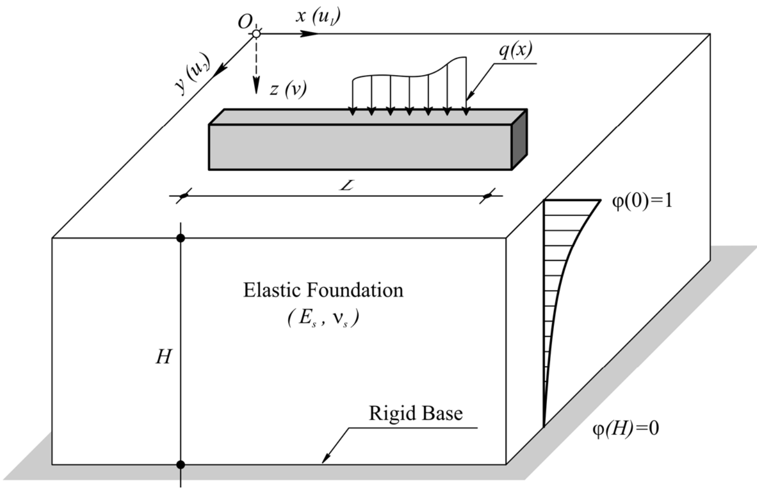


Fig. 3 – Beam resting on two-parameters Vlasov foundation.

2. Governing Differential Equation for Beams on Two-Parameters Elastic Foundation

All foundation models shown foregoing lead to the same differential equation. Basically, all these models are mathematically equivalent and differ only in the definition of their parameters [9]. The various two-parameters elastic foundation models define the reactive pressure of the foundation, $p(x)$, as [9]

$$p(x) = k_s B w(x) - k_1 B \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = k w(x) - \bar{k}_1 \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \tag{4}$$

where: B is the width of the beam cross section; w – deflection of the centroidal line of the beam; k_1 – the second foundation parameter with a different definition for each foundation model. As a special case, if the second parameter

k_1, is neglected, the mechanical modelling of the foundation converges to the Winkler formulation.

Using the last relation and the beam theory, one can generate the governing differential equations for the centroidal line of the deformed beam resting on two-parameters elastic foundation as [9]

$$EI\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4}+k(x)w(x)-\bar{k}_1(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}=q(x)\,, \tag{5}$$

where: E is the modulus of elasticity for the constitutive material of the beam; I – the moment of inertia for the cross section of the beam; $q(x)$ – the distributed load on the beam.

2.1. Parameters Estimation

It is difficult to interpret exactly what subgrade material properties or characteristics are reflected in the various mechanical elements (springs, shears layers, etc.), thus evaluation on a rational, theoretical basis is cumbersome. The advantage of simplified-continuum approach is the elimination of the necessity to determine the values of the foundation parameters, arbitrarily, because these values can be computed from the material properties (deformation modulus, E_s, Poisson number, v_s, and depth of influence zone, H, along the beam) for the soil. Thus there is insight into exactly what each model assumes and implies in terms of subgrade behavior.

Therefore, with the assumptions of vertical displacement,

$$v(x,z)=w(x)\varphi(z)\,, \tag{6}$$

and horizontal displacement,

$$u_1(x,z)=0\,, \tag{7}$$

using variational calculus, Vlasov model parameters are expressed as [2], [7], [8]

$$k_s=\int_0^H \frac{E_s(1-v_s)}{(1+v_s)(1-2v_s)}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right)^2\mathrm{d}z\,,\; k_1=\int_0^H \frac{E_s}{2(1+v_s)}\varphi^2\,\mathrm{d}z\,, \tag{8}$$

where

$$\varphi(z)=\frac{\sinh\gamma\left(1-\dfrac{z}{H}\right)}{\sinh\gamma} \tag{9}$$

is a function defining the variation of the deflection, $v(x, z)$, in the z-direction, which satisfy the boundary condition shown in Fig. 3, and

$$\left(\frac{\gamma}{H}\right)^2 = \frac{1-2\nu_s}{2(1-\nu_s)} \cdot \frac{\int_{-\infty}^{+\infty} \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x}{\int_{-\infty}^{+\infty} w^2 \, \mathrm{d}x} . \tag{10}$$

Since γ is not known *a priori*, the solution technique for parameters evaluation is an iterative process which depends upon the value of the parameter γ. Therefore, by assuming an approximate value of γ initially, the values of k_s and k_1 are evaluated using eq. (8). From the solution of the deflection of the beam, the value of γ is computed using eq. (10). The new γ value is again used to compute new values of k_s and k_1. The proceeding is repeated until two succesive values of γ are approximately equal [7], [8].

3. Example

Even though the continuum mechanics approach may look laborious and difficult to use for a closed-form solution, the numerical model is quite simple and can be easily implemented in application-specific software. An example of such software is EBBEF2p [6] which is developed in MATLAB environment and can handle a wide range of static loading problems involving one-dimensional beams supported by one- or two-parameters elastic foundation, for any loading and boundary condition. Using EBBEF2p computer code, the results for a sample beam resting on Vlasov Simplified Continuum (VSC) model are compared with solutions obtained by two-dimensional finite element plane strain analyses (2-D FEM) (Fig. 4).

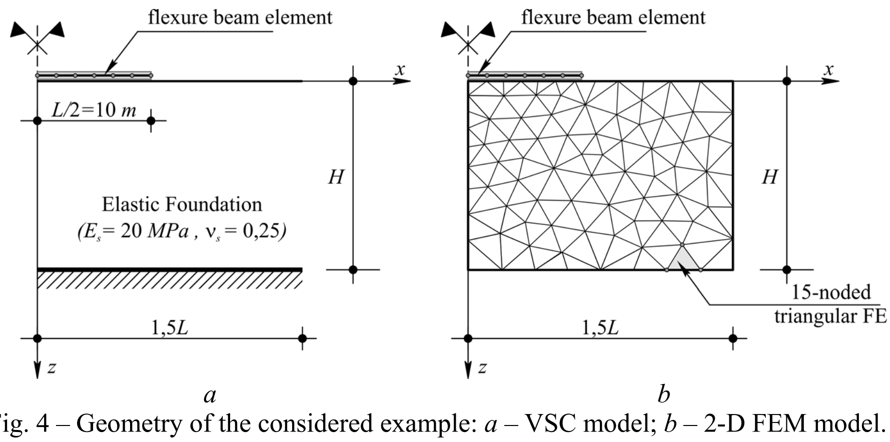


Fig. 4 – Geometry of the considered example: *a* – VSC model; *b* – 2-D FEM model.

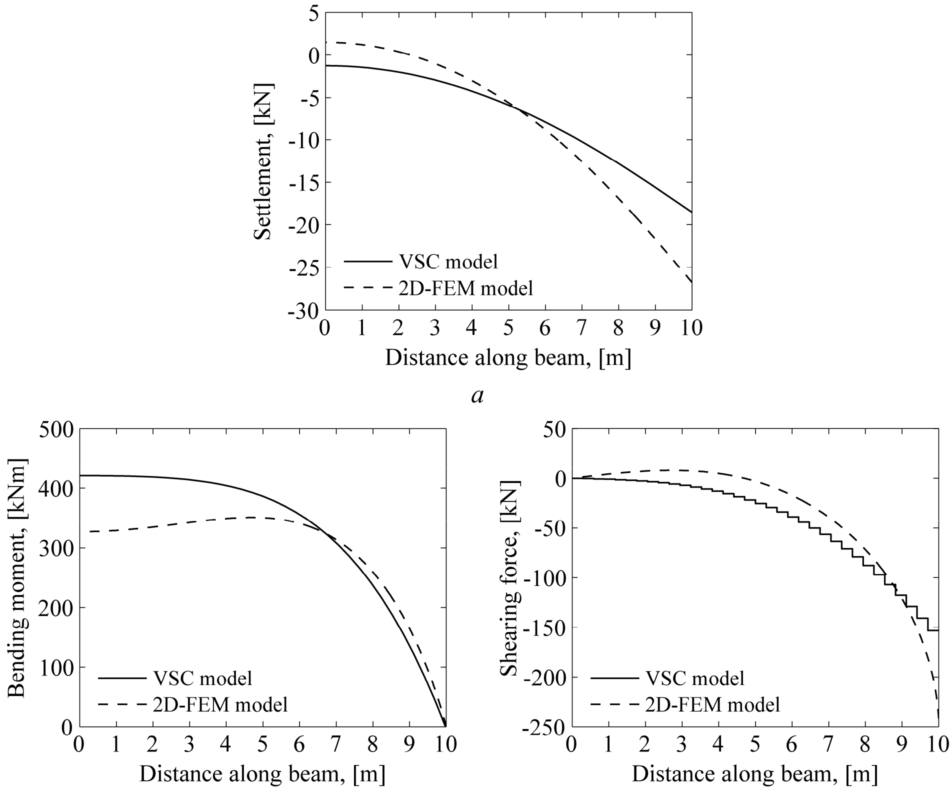
A beam of length L = 20 m, width b = 0.5 m and height h = 1.0 m, with modulus of elasticity E = 27,000 MPa, is considered to be suported by foundation having depth H = 5 m, deformation modulus E_s = 20 MPa and Poissons ratio, ν_s = 0.25 (Fig. 4). The beam carries concentrated loads at ends, 250 kN each.

A total of 905, 15-noded triangular elements with a fourth order interpolation for displacements and twelve Gauss points for the numerical integration were used to define the mesh for the 2-D FEM model.

In both VSC and 2-D FEM models, the beam is modelled with flexure beam element (Table 1).

Table 1
Beam Modelling

	VSC model	2-D FEM model
Element type	linear with 2 nodes	linear with 2 nodes
Total number of nodes	35	469
Total number of elements	34	117



b *c*

Fig. 5 – VSC *vs.* 2-D FEM solution: *a* – settlement; *b* – bending moment; *c* – shearing force.

The results from both 2-D FEM and EBBEF2p technique are shown for comparison in Fig. 5. It can be noted that both solution have almost the same shape and they are in good agreement. However, a full comparison between these two techniques is not fair, because in the 2-D finite element solution, complete compatibility of displacements at the beam–soil interface is assumed, but only vertical displacement compatibility exists in Vlasov model [8].

The results of the final computed values of the soil parameters are presented in Table 2. This demonstrate the versatility of the (Vlasov) simplified continuum foundation model: solve beam on elastic foundation problems without having a need to establish the values of foundation parameters.

Table 2
The Obtained Values of Vlasov Foundation Parameters

k, [kN/m^2]	$\bar{k}_1$, [kN]	γ	Number of iterations
2,437.24	5,953.29	0.953	4

4. Conclusions

The simplified continuum approach, by Vlasov model viewpoint, for static structural analysis of foundation beams, is presented. To demonstrate the versatility of the Vlasov Simplified Continuum (VSC) foundation model, the results for a sample beam resting on VSC model are compared with solutions obtained by two-dimensional finite element plane strain analyses (2-D FEM). As a general observation, the obtained VSM solution are reasonably close to those from more sophisticated finite element solutions.

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"Gheorghe Asachi" Technical University of Iaşi,
Department of Transportation Infrastructure and Foundations
e-mail: bteodoru@ce.tuiasi.ro

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GRINZI REZEMATE PE MEDIU ELASTIC
Abordarea din perspectiva simplificată a mecanicii continuului deformabil

(Rezumat)

Aspectul cheie în proiectarea elementelor structurale rezemate sau în contact cu masivele de pământ se referă la modalitatea prin care intervine în analiză presiunea reactivă a terenului, *p*, ca o încărcare de intensitate şi distribuţie compatibile cu potenţialul masivului de pământ de a prelua încărcări (cazul fundaţiilor izolate sau metodelor simplificate pentru calculul grinzilor de fundare), sau ca o variabilă necunoscută, prin încorporarea sa într-o relaţie conceptuală de proporţionalitate cu tasările fundaţiei (cazul masivelor de pământ idealizate printr-o serie de resorturi a căror constantă de rigiditate este tocmai raportul în care se găsesc presiunile reactive şi tasările – modelul Winkler). Pentru a elimina presiunea reactivă a terenului, ca o variabilă necunoscută în soluţia problemei, se introduce o abordare din perspectiva simplificată a mecanicii continuului deformabil. Comparativ cu modelul Winkler sau alte modele mecanice, idealizarea prezentată prezintă avantajul eliminării necesităţii determinării, în mod arbitrar, a parametrilor caracteristici modelului avut în vedere; aceştia sunt calculaţi în funcţie de caracteristicile de material ale masivului de pământ (modulul de deformaţie liniară, E_s, coeficientul lui Poisson, ν_s) şi adâncimea zonei de influenţă, *H*. Performanţa şi acurateţea formulării, în descrierea răspunsului ansamblului grindă–masiv de pământ, sunt testate prin compararea rezultatelor cu soluţiile obţinute pe modele numerice mai complexe, în ipoteza comportării liniar-deformabile a masivului de pământ.