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SOME ASPECTS REGARDING GOVERNING EQUATIONS OF THIN ANISOTROPIC SHALLOW SHELLS

BY

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Abstract. Following a procedure similar with those used in case of isotropic shallow shells, the governing equations for membrane and bending state of thin anisotropic shallow shells are derived. For a convenient formulation of governing equations, Hooke's law is used in the form of resultants stress–strain relations for the entire multy-layered shell, which can be derived on basis of the formulation of the k layer (lamina). The equilibrium equations and kinematic relations are also used. In the most general case of stress–strain state, a coupling between membrane and bending state exists. The obtained equations are particularized for laminated composite shallow shells having an unsymmetrical stacking sequence, made of orthotropic cross-plied layers. In this case the two stress states are uncoupled. Each of the resolvent equations contain two unknowns: the in-plane force resultants function, F, and w – the displacement perpendicular on the middle surface of the shell. The system of resolvent equations can be solved by analytic way, using the Fourier series developments of unknown functions and corresponding boundary conditions.

Key words: shallow shell; anisotropic material; membrane and bending state; tensile, coupling and bending stiffnesses; governing equations.

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1. Introduction

The concept of shallow shell refers to the shells with a large curvature radius (*i.e.* small curvature). One of the criteria used in classifying the shells is the ratio between the rise of the vault and the maximum size in plane. From this point of view shells may be high (or deep), respectively shallow. Most of the researchers include in the category of shallow shells those that satisfy the condition (Soare, 1968; Beleş & Soare, 1969; Quatu, 2004):

$$\frac{f}{L_{1,2}} < \frac{1}{5}, \tag{1}$$

where: f is the maximum rise of the middle surface towards the base plane surface passing through the support's points of the shell; L_1 , L_2 – the maximum/minimum distance between the supports (or the dimensions of the shell following the direction of the coordinates lines).

In the case of reinforced and prestressed concrete shells there are some regulations where the ratio from eq. (1) is limited to 1/4 (Indicativ NP *119*/2006). Also for shells having a hyperbolic paraboloide shape, in Mileikovski & Kupar, 1978, the conditions of shallowness, which identify the internal geometry of the surface with plane coordinate geometry, are presented. Internal geometry of the surface is characterized by the line's length and the angle's values between them.

In Romania there is no monograph exclusively dedicated to shallow shells, less to anisotropic shallow shells. Chapters or short references to the analysis of stress and strain state of isotropic shallow shells can be found in Soare, 1968; Beleş & Soare, 1969; Mihăilescu, 1977; Cioclov, 1983. Aspects of steel shallow shells' stability are discussed in Pavel, 1985. Non-linear equations of smooth or stiffened with ribs thin shallow shells are discussed in Ginocu & Ivan, 1978.

Internationally can be found a vast literature dedicated to anisotropic shallow shells. From the numerous monographs, treatises, papers or research reports we first mention the work of S.A. A m b a r t s u m i a n, that uses the terminology of "extremely shallow shells" (Ambartsumian, 1956; Ambartsumian, 1991).

Shallow shells can also be classified in thin or thick, therefore there are two theories for shallow shells (Quatu, 2004).

The first is a classical shallow shell theory (Classical Shallow Shell Theory – CSST), and the second is a shear deformation shallow shell theory (Shear Deformation Shallow Shell Theory – SDSST). None of them offers a reduction regarding the equation's degree of the shell in comparison with the general theories of shell, but they do significantly reduce the terms and lead to a simplification of the equations.

Shallow shells may have different types of curvature (cylindrical circular, spherical, conical, elliptical, hyperbolic paraboloide surface, etc.) as well as different types of planforms (rectangular, triangular, trapezoidal, circular, elliptical, etc.).

The appearance and development of new materials (especially of laminated composites materials), new constructive technologies and systems of making shallow shells implies a continuous effort of perfection and adaptation of theories and computation methods. Reducing the complexity of the equations as well as the improvement and diversification of the analytical or numerical methods to obtain the solutions for the static/dynamic analysis (free or forced vibrations) and of stability for anisotropic shallow shells, still make the subject of modern research (Wang & Schweizerhof, 1995; Piskunov *et al.*, 2001; Semenyuk & Trach, 2010).

The present work is part of this context and its objectives are systematizations and particularizations of the bending theory for thin anisotropic shallow shells made of composite materials.

2. Fundamental Equations of Thin Shallow Shells

2.1. Primary Hypotheses

A shallow shell is characterized by its middle surface defined by the equation

$$z = -\frac{1}{2}\left(\frac{\alpha^2}{R_\alpha} + 2\frac{\alpha\beta}{R_{\alpha\beta}} + \frac{\beta^2}{R_\beta}\right), \tag{2}$$

where: α, β are the directions of the curvilinear coordinates; R_α , R_β , $R_{\alpha\beta}$ – the radii of curvature in α, β directions and, respectively, the torsion's curvature radius.

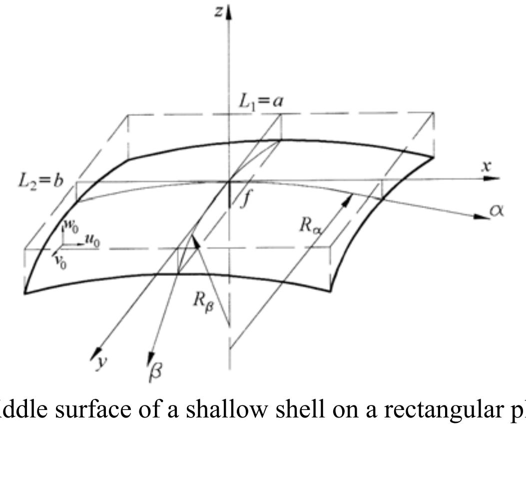

Fig. 1 – Middle surface of a shallow shell on a rectangular planform.

A middle surface described by eq. (2) is presented in Fig. 1 for positive values of the radii of curvature.

The theory of shallow shells is based on three supplementary hypotheses in comparison to the classical theory of shells namely

1. The radii of curvature are very large compared to the inplane displacements, u and v; also the transverse shearing force are smaller than the $R_i(\partial N_i / \partial i)$ terms

$$\frac{u_i}{R_i} = 1; \quad V_i = R_i \frac{\partial N_i}{\partial i}, \tag{3}$$

where u_i is u or v, V_i is any of the shear forces V_α and V_β, R_i is R_α, R_β or $R_{\alpha\beta}$, N_i is N_α, N_β or $N_{\alpha\beta}$. The term ∂i indicates the derivative related to any of the variables α or β.

2. The term z/R_i may be neglected with respect to 1.

3. The shell is shallow enough to be represented in the coordinates system in the plane (the metric of the surface can be replaced with the metric from in plane).

A differential element of arch from the surface is given by the relation

$$(\mathrm{d}s)^2 = A^2(\mathrm{d}\alpha)^2 + B^2(\mathrm{d}\beta)^2, \tag{4}$$

where the Lamé parameters of the surface, A and B, are constant in the case of rectangular orthotropy.

The above mentioned hypotheses, together with those of thin shell from the classical theory of shells, allow the writing of differential equilibrium equations and of geometric equations in orthogonal Cartesian coordinates.

2.2 The Equilibrium Differential Equations

A differential element is detached from the shallow shell, with parallel planes to the planes of coordinates, and the following stress resultants forces and moments are introduced on it (Fig. 2):

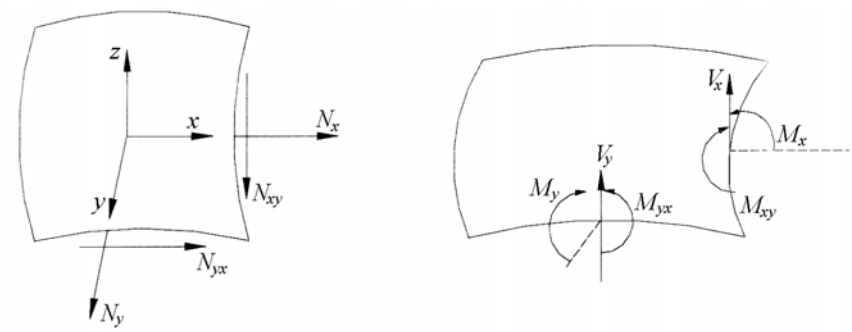

Fig. 2 – Differential element with stress resultants force and moments.

a) membrane stress resultants (N_x , N_y – tension and compression normal forces and $N_{xy} = N_{yx}$ – shear forces from the tangent plane to the middle surface);

b) transverse stress resultants (M_x , M_y – bending moments, $M_{xy} = M_{yx}$ – twisting moments; V_x , V_y – transverse shear forces).

The equilibrium conditions of the differential element of the shell, under the action of sectional stress resultants and of surface and massic forces, lead to the following system of eqs. (Soare, 1968):

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}+X=0 \quad (a); \quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}+Y=0 \quad (b);$$

$$rN_x+2sN_{xy}+tN_y+\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=pX+qY-Z \quad (c); \qquad (5)$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}-V_x=0 \quad (d); \quad \frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-V_y=0 \quad (e),$$

where X, Y are the components of the body forces along the axis from the tangent plane, and Z is the density of the body force and from the surface in the normal direction (usually, the gravitational direction).

It must be noted that the inertia forces were not included so the equations refer to static equilibrium.

Eqs. (5 *a*) and (5 *b*) are of the membrane state, eqs. (5 *d*) and (5 *e*) of the bending state and eq. (5 *c*) is the coupling equation between membrane and bending state. In this equation the notations of Monge were used (Soare, 1968; Beleş & Soare, 1969)

$$p=\frac{\partial z}{\partial x}, \quad q=\frac{\partial z}{\partial y}, \quad r=\frac{\partial^2 z}{\partial x^2}, \quad s=\frac{\partial^2 z}{\partial x \partial y}, \quad t=\frac{\partial^2 z}{\partial y^2}, \qquad (6)$$

which represent the first and quadratic order derivative of middle surface's equation, given explicitly in the form $z = f(x, y)$.

In the case of shallow shells r, t are normal curvatures and s is the torsion curvature. They results directly from the equation of middle surface.

Eqs. (1 *a*) and (1 *b*) are satisfied if the stress resultants forces, N_x , N_y , N_{xy} , are expressed by a function $F(x, y)$, which generates these forces

$$N_x=\frac{\partial^2 F}{\partial y^2}-\int X\mathrm{d}x \quad (a); \quad N_y=\frac{\partial^2 F}{\partial x^2}-\int Y\mathrm{d}y \quad (b); \quad N_{xy}=-\frac{\partial^2 F}{\partial x \partial y} \quad (c). \quad (7)$$

The stress resultant forces, N_x , N_y , N_{xy} , with their expressions from (7), will be replaced in eq. (5 *c*). Also the derivatives of the shear forces from the same equation can be expressed in relation with the derivatives of the moments, using eqs. (5 *d*) and (5 *e*), and then they are replaced in eq. (5 *c*) which becomes

$$t\frac{\partial^2 F}{\partial x^2}-2s\frac{\partial^2 F}{\partial x\partial y}+r\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=-P\left(x,y\right), \qquad (8)$$

where $P(x, y)$ is an external loading function

$$-P\left(x,y\right)=pX+qY-Z+r\int X\mathrm{d}x+t\int Y\mathrm{d}y. \qquad (9)$$

2.3. Kinematic Relations

The relations between the strains from membrane state, ε_x, ε_y, γ_{xy}, and the displacements *u, v* from the tangential plane, respectively *w* in normal direction to the middle surface of the shallow shell, have the following form:

$$\varepsilon_x=\frac{\partial u}{\partial x}-rw \quad (a); \quad \varepsilon_y=\frac{\partial v}{\partial y}-tw \quad (b); \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2sw \quad (c). \qquad (10)$$

First the derivatives $\partial^2\varepsilon_x/\partial y^2$, $\partial^2\varepsilon_y/\partial x^2$, $-\partial^2\gamma_{xy}/\partial x\partial y$ should be calculated and then their addition should be calculated too. A compatibility equation results, where the displacements *u* and *v* are no longer introduced

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=-\left(t\frac{\partial^2 w}{\partial x^2}-2s\frac{\partial^2 w}{\partial x\partial y}+r\frac{\partial^2 w}{\partial y^2}\right). \qquad (11)$$

Eqs. (8) and (11) can be applied in the case of shallow shells made of elastic material with the constant curvatures *r*, *t*, *s*.

Further we can utilize different elastic materials using the appropriate constitutive equations.

2.4. Constitutive Equations

The case of shells made of anisotropic materials with ordinate anisotropy (laminated composites) will be discussed.

The usual hypotheses for laminated composites consider each layer (lamina) made of parallel fibers as incorporated in a matrix material. Thus the material of each layer can be considered macroscopically homogeneous, orthotropic and linear elastic. Also, it is supposed that the fibers from each layer follow the coordinates of the shallow shell or they form a constant angle with these lines of coordinates.

The most general case is that of the materials that have a coupling between the membrane state and bending state, so that between stress resultants and strains the following type relations are made:

$$\{N\}=[A]\{\varepsilon^0\}+[B]\{k\} \quad (a); \quad \{M\}=[B]\{\varepsilon^0\}+[D]\{k\} \quad (b), \qquad (12)$$

where

$$\begin{aligned}\{N\}&=\{N_x \quad N_y \quad N_{xy}\}^T, \quad \{M\}=\{M_x \quad M_y \quad M_{xy}\}^T \quad (a);\\ \{\varepsilon^0\}&=\{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\}^T, \quad \{k\}=\{k_x \quad k_y \quad k_{xy}\}^T \quad (b);\end{aligned} \qquad (13)$$

$\{N\}$, $\{M\}$ are vectors of the in-plane force resultants (membrane state), respectively of the moment resultants (bending state); $\{\varepsilon^0\}$ – vector of the membrane strain from the middle surface; $\{ k \}$ – vector of the curvature changes of the middle surface; $[A]$, $[B]$, $[D]$ – matrices of tensile, coupling and bending stiffnesses, having the following expressions:

$$[A]=\begin{bmatrix}A_{11} & A_{12} & A_{16}\\ A_{12} & A_{22} & A_{26}\\ A_{16} & A_{26} & A_{66}\end{bmatrix}; \ [B]=\begin{bmatrix}B_{11} & B_{12} & B_{16}\\ B_{12} & B_{22} & B_{26}\\ B_{16} & B_{26} & B_{66}\end{bmatrix}; \ [D]=\begin{bmatrix}D_{11} & D_{12} & D_{16}\\ D_{12} & D_{22} & D_{26}\\ D_{16} & D_{26} & D_{66}\end{bmatrix}. \quad (14)$$

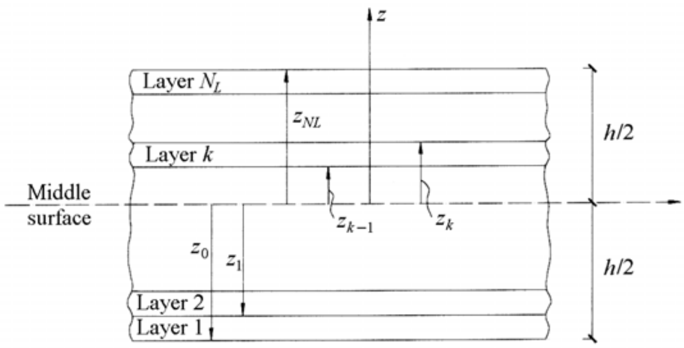

Fig. 3 – Cross-sectional view of the multy-layered shell.

The coefficients A_{ij}, B_{ij}, D_{ij} of the membrane, coupling and bending global stiffnesses, depend on the reduced stiffnesses of the laminae, $\left[\overline{Q}_{ij}\right]$, and on their positions related to the middle surface of the laminate. They results by the sequential integration layer by layer on the thickness, h, of the laminate composed of N_L layers

$$A_{ij} = \sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)} \left(z_k - z_{k-1}\right);\ B_{ij} = \frac{1}{2}\sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)} \left(z_k^2 - z_{k-1}^2\right);\ D_{ij} = \frac{1}{3}\sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)} \left(z_k^3 - z_{k-1}^3\right), \quad (15)$$

where z_k is the distance from the middle surface to the surface of layer k with the most distant coordinate z (Fig. 3).

For an orthotropic layer, the stress–strain relations can be expressed in terms of layer's fiber directions (or local coordinates 1 and 2) as (Fig. 4)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}, \quad (16)$$

or in a compact form:

$$\{\sigma\} = [Q]\{\varepsilon\}. \quad (17)$$

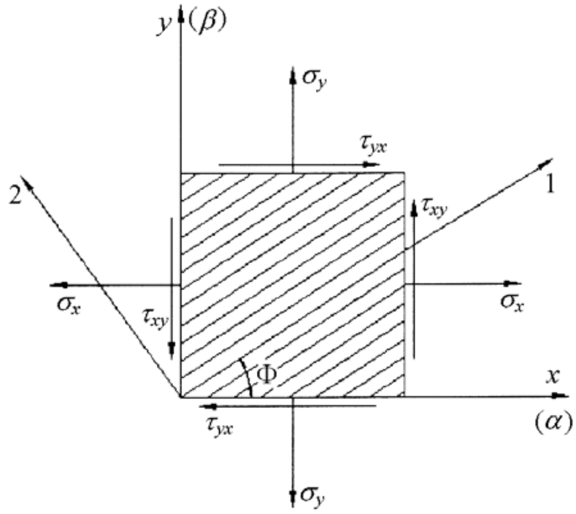

Fig. 4 – Local (1, 2) and global (x, y) coordinates of fiber reinforced material.

The stress–strain relations for a typical lamina in terms of laminated composite shell coordinates become

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \tag{18}$$

or in a more compact presentation

$$\{\sigma\} = \left[\overline{Q}\right]\{\varepsilon\}. \tag{19}$$

The constants $\overline{Q}_{ij}$ are the elastic stiffness coefficients, which are found from eq.

$$\left[\overline{Q}\right] = [T]^{-1}[Q][T], \tag{20}$$

where the transformation of stresses from local coordinates 1, 2 of the lamina to the global shell coordinates (*α, β* or *x, y*) can be performed using the transformation matrix

$$[T] = \begin{bmatrix} \cos^2\varphi & \sin^2\varphi & 2\cos\varphi\sin\varphi \\ \sin^2\varphi & \cos^2\varphi & -2\cos\varphi\sin\varphi \\ -\cos\varphi\sin\varphi & \cos\varphi\sin\varphi & \cos^2\varphi - \sin^2\varphi \end{bmatrix}. \tag{21}$$

From the constitutive eqs. (12), that are similar to those of thin anisotropic plates, it can be noticed that the membrane state is characterized by the force resultants, [*N*], and the strains, $\{\varepsilon^0\}$, connected through matrix [*A*], and the bending state by moment resultants, [*M*], and curvature changes, [*k*], connected by matrix [*D*]. The connection between the two tensional states is realized by the coupling matrix [*B*].

In the particular case of the symmetric laminates regarding the middle surface of the shell, the uncoupling between the two tensional states is realized because the coupling matrix [*B*] becomes null resulting

$$\{N\} = [A]\{\varepsilon^0\} \quad (a); \quad \{M\} = [D]\{k\} \quad (b). \tag{22}$$

A more particular case that we'll further discuss is that of cross plied laminates. In this situation the coupling matrix, [*B*], is zero and the shear is uncoupled from tension and the bending from torsion. Matrices [*A*] and [*D*] become

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \quad (a); \quad [B] = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \quad (b). \tag{23}$$

Between the stress force resultants and the membrane strains the following relation exists:

$$\begin{Bmatrix} N_1 = N_x \\ N_2 = N_y \\ N_6 = N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 = \varepsilon_x \\ \varepsilon_2^0 = \varepsilon_y \\ \varepsilon_6^0 = \gamma_{xy} \end{Bmatrix}, \tag{24}$$

or the inverse form:

$$\begin{Bmatrix} \varepsilon_1^0 = \varepsilon_x \\ \varepsilon_2^0 = \varepsilon_y \\ \varepsilon_6^0 = \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{A_{22}}{A_{11}A_{22} - A_{12}^2} & -\dfrac{A_{12}}{A_{11}A_{22} - A_{12}^2} & 0 \\ -\dfrac{A_{12}}{A_{11}A_{22} - A_{12}^2} & \dfrac{A_{22}}{A_{11}A_{22} - A_{12}^2} & 0 \\ 0 & 0 & \dfrac{1}{A_{66}} \end{bmatrix} \begin{Bmatrix} N_1 = N_x \\ N_2 = N_y \\ N_6 = N_{xy} \end{Bmatrix}. \tag{25}$$

The notations used in technique,

$$\begin{gathered} E_1 = E_x = \frac{A_{11}A_{22} - A_{12}^2}{hA_{22}}; \quad E_2 = E_y = \frac{A_{11}A_{22} - A_{12}^2}{hA_{11}}; \\ \nu_{21} = \nu_{yx} = \frac{A_{12}}{A_{11}}; \quad \nu_{12} = \nu_{xy} = \frac{A_{12}}{A_{22}}; \quad G_{12} = G_{xy} = \frac{A_{66}}{h}, \end{gathered} \tag{26}$$

are introduced, where: E_i, G_{ij}, ν_{ij}, (i = 1, 2 or $i = x, y$) are moduli of longitudinal elasticity (Young's moduli), shear moduli and Poisson's coefficients of the composite laminated material with h thickness.

With the notations from the technique of the elastic constants, the strains from eq. (25) can be written according to the membrane force resultants,

$$\varepsilon_x = \frac{N_x}{E_x h} - \frac{\nu_{yx} N_y}{E_y h}; \quad \varepsilon_y = -\frac{\nu_{xy} N_x}{E_x h} + \frac{N_y}{E_y h}; \quad \gamma_{xy} = \frac{N_{xy}}{G_{xy} h}. \tag{27}$$

In the case of the analysed material, the uncoupling extends between the normal stresses from traction, respectively bending and tangential stresses from shear, respectively torsion. Eq. (22 *b*) taking into account (13 *b*) and (23 *b*), becomes

$$\begin{Bmatrix} M_1 = M_x \\ M_2 = M_y \\ M_6 = M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}, \tag{28}$$

or in algebraic form

$$M_x = D_{11}k_x + D_{12}k_y, \quad M_y = D_{12}k_x + D_{22}k_y, \quad M_{xy} = D_{66}k_{xy}, \tag{29}$$

where

$$k_x = -\frac{\partial^2 w}{\partial x^2}; \quad k_y = -\frac{\partial^2 w}{\partial y^2}; \quad k_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}. \tag{30}$$

3. Resolvent Equations and Solutions

The expressions of the moments from (29) are derived and introduced in (8). Also the expressions of strains from (27) are derived and they are introduced in the compatibility eq. (11). After calculation and systematization we obtain two differential equations in the unknowns F and w namely

$$t\frac{\partial^2 F}{\partial x^2} - 2s\frac{\partial^2 F}{\partial x \partial y} + r\frac{\partial^2 F}{\partial y^2} - D_{11}\frac{\partial^4 w}{\partial x^4} - 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} - D_{22}\frac{\partial^4 w}{\partial y^4} = -Z, \tag{31}$$

$$\begin{gathered} \frac{1}{E_y}\frac{\partial^4 F}{\partial x^4} - \left(\frac{\nu_{xy}}{E_x} + \frac{\nu_{yx}}{E_y} - \frac{1}{G_{xy}}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x}\frac{\partial^4 F}{\partial y^4} + \\ + h\left(t\frac{\partial^2 w}{\partial x^2} - 2s\frac{\partial^2 w}{\partial x \partial y} + r\frac{\partial^2 w}{\partial y^2}\right) = 0. \end{gathered} \tag{32}$$

In the right side of (31) only the gravitational component of the load, Z, was kept.

The integration of this system of differential equations with partial derivatives can be performed from an analytic point of view, using solutions based on expansions in Fourier series, for particular cases of boundary conditions. The solution can be taken under the form

$$\begin{aligned} F(x,y) &= \sum_m \sum_n F_{mn}^0 \cos(\alpha_m x)\cos(\alpha_n y) \quad (a), \\ w(x,y) &= \sum_m \sum_n w_{mn}^0 \sin(\alpha_m x)\sin(\alpha_n y) \quad (b), \end{aligned} \tag{33}$$

where

$$\alpha_m = \frac{m\pi}{a}, \quad \alpha_n = \frac{n\pi}{b}, \quad (m, n = 1, 2, 3...), \tag{34}$$

and the domain occupied by the shallow shell is rectangular, with *a* and *b* sides parallel with the coordinate axis *x* and *y*.

The solutions (33) must satisfy the boundary conditions, from which results the coefficients, F_{mn}^0, respectively, w_{mn}^0. The boundary conditions also include derivatives of these functions, resulting from the supporting types of the shallow shell on the contour. Numerical solutions of the system can also be obtained using the finite difference method.

4. Conclusions

The specific hypotheses of the shallow shell's theory lead to the reduction of some terms and the simplifying of resolvent equations, without reducing the degree of these equations compared to those from the general theory of shells.

Material or structural anisotropy, present at shells of laminated composites, represents a difficulty factor in establishing the resolvent equations and their solutions. The complexity of the equations is increased by the anisotropic coupling between the membrane's stresses and the bending's stresses through the coupling matrix $[B] = [B_{ij}]$.

In particularly cases of arrangement of the layers, as in the case of symmetric distribution from the middle surface of the laminate, matrix $[B]$ becomes null and so the stress resultants from the two tensional states are uncoupled. The same happens with non-symmetrical laminates but with cross-plied fibers. For this stacking sequence of lamination, following the path from isotropic shells, two resolvent equations were obtained, having as unknowns a function of stress resultants, *F*, and the function of normal displacement, *w*. In eq. (31) the bending stiffnesses, D_{ij}, $(i, j = 1, 2, 6)$, of the laminate appear and in eq.(32) the technical elastic constants of an equivalent monolayer orthotropic material appear. The computation relations of these characteristics according to the elastic constants of lamellae's components were deduced.

The obtained system of resolvent equations can be analytically solved based on Fourier expansions, or numerically, using the finite difference method.

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UNELE ASPECTE PRIVIND ECUAȚIILE DE GUVERNARE ALE PLĂCILOR CURBE SUBȚIRI ANIZOTROPE PLEOŞTITE

(Rezumat)

Urmând un procedeu similar celui utilizat la plăci curbe izotrope pleoştite, se deduc ecuațiile de guvernare pentru starea de membrană şi de încovoiere a plăcilor curbe subțiri pleoştite anizotrope. Pentru o formulare convenabilă a ecuațiilor de guvernare este utilizată legea lui Hooke sub forma relațiilor dintre rezultantele tensiunilor şi deformații pentru întreaga învelitoare multistrat, care pot fi determinate pe baza formulării pentru un strat (lamină) oarecare, *k*. Sunt utilizate, de asemenea, ecuațiile de echilibru şi relațiile cinematice. În cel mai general caz al stării de tensiune–deformație, există o cuplare între starea de membrană şi starea de încovoiere. Ecuațiile obținute sunt particularizate pentru învelitori pleoştite compozite laminate, având o secvență de stivuire nesimetrică, realizate din straturi ortotrope încrucişate. În acest caz cele două stări de tensiune sunt decuplate. Fiecare dintre ecuațiile rezolvente conține două necunoscute: funcția de eforturi din plan, *F*, şi *w* – deplasarea normală pe suprafața mediană a învelitorii. Sistemul de ecuații rezolvente poate fi soluționat pe cale analitică, utilizând dezvoltările în serie Fourier ale funcțiilor necunoscute şi condiții pe contur corespunzătoare.