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EFFECT OF CONTIGUITY ON SHEAR ELASTIC MODULUS OF FIBRE REINFORCED POLYMERIC COMPOSITES

BY

RALUCA HOHAN[1], LILIANA BEJAN[2,*] and NICOLAE ŢĂRANU[1]

“Gheorghe Asachi” Technical University of Iaşi
[1]Department of Civil Engineering
[2]Department of Theoretical Mechanics

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Abstract. Unidirectional fibre reinforced polymer (FRP) composite elements are utilized nowadays in various applications in engineering structures. Many composite structures are made of unidirectionally reinforced laminas stacked in a certain pattern that matches the required stiffness and strength performance criteria. The constitutive equations for orthotropic laminas require the elastic constants associated with principal material axes. The elastic shear modulus of a composite lamina is a matrix dominated property that can be determined analytically using the inverse rule of mixtures. However the results obtained in such a way are much lower than the experimental values and some improvements are needed to develop more credibile evaluation models. The paper presents the most appropriate models and, in particular, the influence of contiguity factor, taking into account the effective fibre volume fractions in various fibre arrays.

Key words: fibrous composites; elastic constants; shear modulus; contiguity factor; analytical models.

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[*]Corresponding author *e-mail*: lilbejan@yahoo.com

1. Introduction

Fibre reinforced polymer (FRP) composites consist of fibres of high hardness, strength and modulus embedded in a softer and weaker matrix, with distinct interfaces between them. Both constituents retain their physical and chemical identities, but their combination leads to properties that cannot be achieved with either of components working individually (Mallick, 2008).

In the case of FRP composites the reinforcing fibres constitute the backbone of the material and they determine most of its strength and stiffness in the direction parallel to fibres. The polymeric matrix binds together the fibres and protects their surfaces from damage. It disperses the fibres, separates them and also transfers stresses to them.

Most composite structures made of fibrous composites consist of several distinct unidirectional laminas.

A unidirectional composite consists of parallel fibres embedded in a matrix and a lamina is a flat or curved arrangement of unidirectional or woven fibres in a support matrix.

The unidirectional lamina (Fig. 1) is the basic building block in a laminated FRP composite. The direction parallel to the fibres is called the *longitudinal direction* (axis 1 or *L*) and the direction perpendicular to the fibres in the 1-2 plane is called the *transverse direction*. Any direction in the 2-3 plane is also a *transverse direction*. These axes are also referred to as the material axes of the lamina.

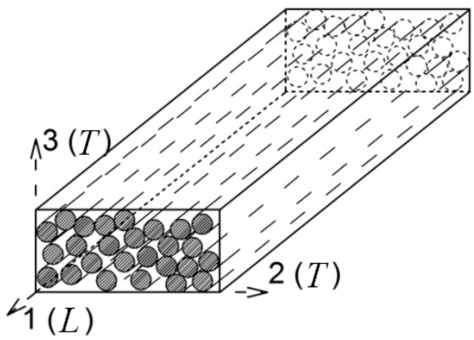


Fig. 1 – A unidirectional fibre reinforced lamina with its principal material axes.

The unidirectional composite shows different properties in the material axes directions. Thus, this type of composites is *orthotropic* with their axes 1, 2, 3 as axes of symmetry.

2. Plane Stresses State and Specially Orthotropic Lamina

A lamina with the reference axes coinciding with the axes of material (Fig. 2) is called *specially orthotropic lamina*; the plane state of stresses, typical to this element, is also illustrated in this figure.

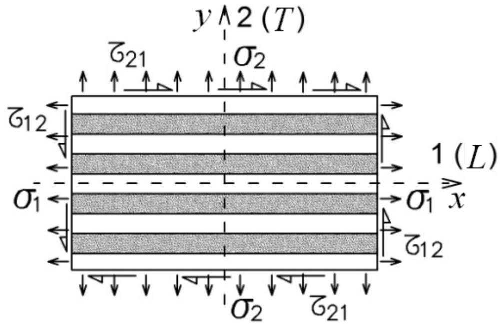


Fig. 2 – Specially orthotropic lamina under plane state of stress.

In the analysis of composite structures it is often the case when a condition of plane stress actually exists or it is a very good approximation. Constitutive eqs. in principal material coordinates of a specially orthotropic lamina for the plane stress state (Tsai *et al*., 1980; Mallick, 2008) are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}, \tag{1}$$

where Q_{ij}, $(i, j = 1, 2, 3)$, are the terms of the reduced stiffness matrix.

It may be pointed out that for two-dimensional orthotropy only four constants are needed namely, E_1, E_2, G_{12} and ν_{12}; these constants can be measured in the laboratory and are termed *engineering constants*.

The expressions for Q_{ij}, in terms of engineering constants, are

$$\begin{gathered} Q_{11} = \frac{E_1}{1-\nu_{21}\nu_{12}}, \quad Q_{12} = \frac{\nu_{21}E_1}{1-\nu_{21}\nu_{12}} = \frac{\nu_{12}E_2}{1-\nu_{21}\nu_{12}}, \\ Q_{22} = \frac{E_2}{1-\nu_{21}\nu_{12}}, \quad Q_{66} = G_{12}, \end{gathered} \tag{2}$$

where E_1 is the longitudinal elastic modulus, along the fibres (1), E_2 – the transverse elastic modulus, perpendicular to the fibres direction (2), G_{12} – the shear modulus of elasticity in the plane of lamina (1, 2), ν_{12} – the major Poisson's ratio and ν_{21} – the minor Poisson's ratio.

The stiffness properties can be determined by experimental measurements, but one set of experimental measurements determines the properties of a fibre-matrix system produced by a single fabrication process. When any change in the system variables occur, additional measurements are required. These experiments may become time consuming and cost prohibitive; therefore, a variety of methods, based on micromechanics, have been used to predict them (Agarwal *et al*., 2006; Daniel *et al*., 2006). In prediction studies, micromechanics mean the analysis of the effective composite properties in terms of constituent material properties. The *E*-glass fibres have been utilized in the analysis presented in this paper. Many FRP composites for structural applications in civil engineering are based on thermosetting polymers. These polymers develop a spatial network that sets them in shape. If they are heated after they have been cured, they do not melt and will retain their shape until they begin to thermally decompose at high temperature (Askeland *et al*., 2010; Hollaway, 2010). An epoxy polymer matrix with elastic modulus $E = 3$ GPa and Poisson's ratio $\nu = 0.38$ has been utilized in this paper for numerical calculations.

3. Geometry of Fibres Distribution and Contiguity

The range of constituent volume fractions that may be expected in fibre reinforced composites can be determined using representative area elements for idealized fibre-packing geometries such as the triangular and square arrays shown in Fig. 3.

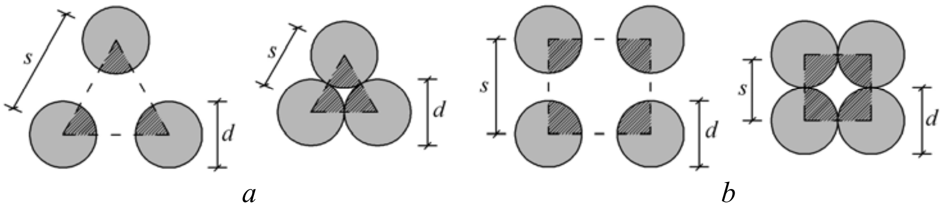


Fig. 3 – Representative area elements for idealized fibre- packing geometries
a – square array and fibres position for $V_{f\,\max}$; *b* – triangular array and fibres position for $V_{f\,\max}$.

If it is assumed that the fibre spacing, *s,* and the fibre diameter, *d*, do not change along the fibre length, then, the area fractions must be equal to the volume fractions (Ţăranu *et al*., 2005) The fibre volume fraction for the square array is found by dividing the area of the fibre enclosed in the square by the total area of square. The maximum theoretical fibre volume fraction in the square area occurs when $s = d$,

$$V_{f\max} = \frac{\pi}{4} = 0.785. \tag{3}$$

In case of a triangular array when $s = d$, the maximum fibre volume fraction is

$$V_{f\max} = \frac{\pi}{2\sqrt{3}} = 0.907. \tag{4}$$

These theoretical limits are not generally achievable in practice. In most continuous fibre composites the fibre volume fractions range from 0.5 to 0.75.

The reinforcing fibres on the cross section of unidirectional composite materials are usually randomly arranged (Fig. 4 *b*) instead of being distributed in a regular array.

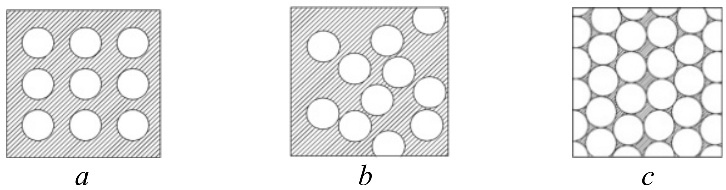


Fig. 4 – Various schemes of fibre contiguity: *a* – isolated fibres ($C = 0$); *b* – actual random fibre ($0 < C < 1$); *c* – isolated matrix ($C = 1$).

Considering C as the degree of contiguity, $C = 0$ corresponds to isolated fibres (Fig. 4 *a*) and $C = 1$ corresponds to all contiguous fibres (Fig. 4 *c*).

4. The Influence of Contiguity on the Shear Elastic Modulus

The behaviour of unidirectional composites under in-plane shear loading is dominated by the matrix properties and the local stress distributions. The mechanics of materials approach uses a *series model under uniform shear stress* (Fig. 5) to determine the shear modulus.

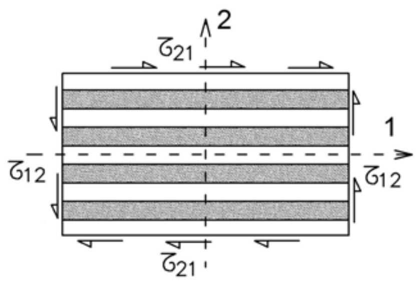


Fig. 5 – The model used for the shear elastic modulus analytical evaluation.

The shear stresses are equal in fibres, matrix and composite and the compatibility of shear deformations is assured (Gibson, 2012). The in-plane shear modulus, G_{12}, determined on the model is defined by relation

$$G_{12} = \frac{\tau_{12}}{\gamma_{12}}, \tag{5}$$

where τ_{12} is average composite shear stress in the (1,2) plane and γ_{12} is the average engineering shear strain in the same plane.

Using the model given in Fig. 5, a formula based on the inverse rule of mixtures has been deduced

$$G_{12} = \frac{G_f G_m}{G_m V_f + G_f V_m}, \tag{6}$$

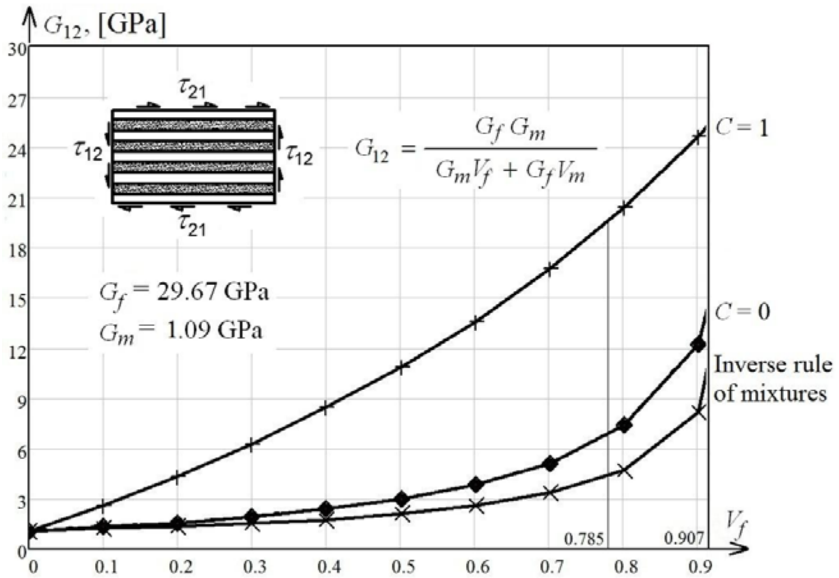


Fig. 6 – Variation of G_{12} by inverse rule of mixtures compared to extreme contiguity.

and the corresponding shear modulus values are illustrated in Fig. 6, the bottom curve, where G_{12} is the in-plane shear modulus of the composite, G_f – the shear modulus of fibres, G_m – the shear modulus of matrix; V_f and V_m – the fibres and matrix volume fractions, respectively.

The utilized model for G_{12} does not give precise results because it is based on many simplifications. In addition, the shear modulus values determined experimentally are significantly higher than those determined with eq. (6). To obtain more accurate results complex analyses have been performed to give convenient predictions (Mathews *et al*., 2008). Some attempts have focussed on improving the inverse rule of mixture formula for shear loading, one of them based on contiguity.

The approach that considers the contiguity has been analysed by Tsai (Jones, 1999) who has developed the following formulas for the shear modulus:

$$G_{12} = \left(1-C\right)G_m \frac{2G_f - \left(G_f - G_m\right)V_m}{2G_m + \left(G_f - G_m\right)V_m} + CG_f \frac{\left(G_f + G_m\right) - \left(G_f - G_m\right)V_m}{\left(G_f + G_m\right) + \left(G_f - G_m\right)V_m}. \quad (7)$$

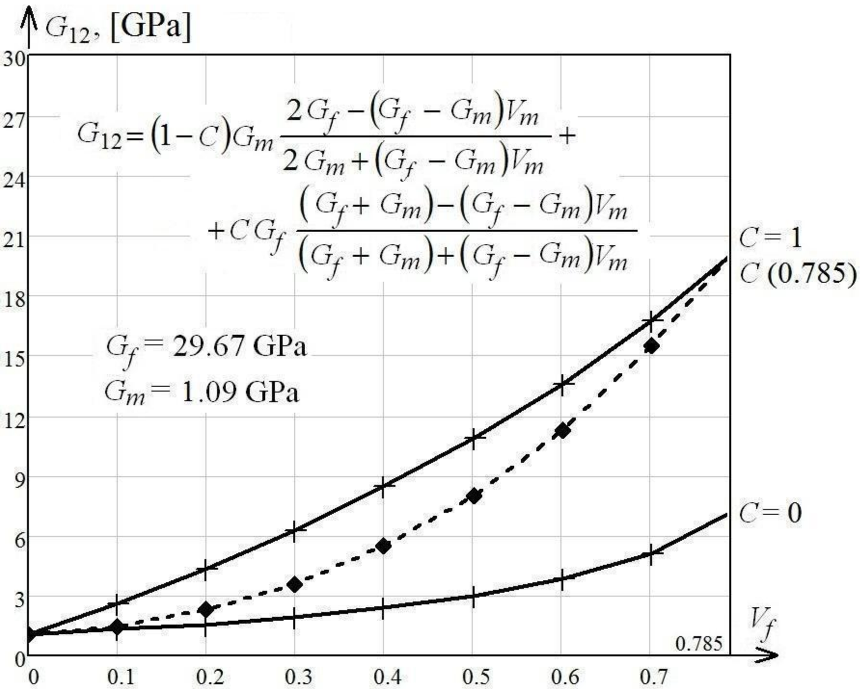
The contiguity factor, C, takes values between 0 and 1 (0 in case of totally separated fibres (Fig. 4 *a*) and 1 for total fibres contiguity (Fig. 4 *c*)). An averaging factor has been used by the authors to determine the intermediate stiffness values, considering the square array of reinforcing fibres ($C_{0.785}$) and the triangular array ($C_{0.907}$) respectively. These averaging factors take into account the effective fibre volume fractions

$$C_{0.785} = \frac{V_f}{0.785}, \quad C_{0.907} = \frac{V_f}{0.907}. \quad (8)$$

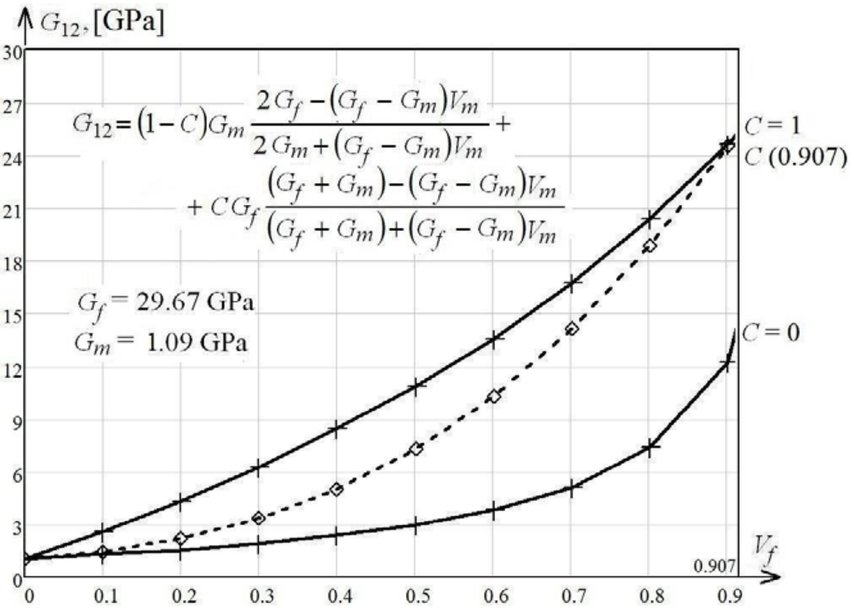
Figs. 7 *a* and 7*b* represent the influence of contiguity on shear modulus, considering the averaging factor effects ($C_{0.785}$ and $C_{0.907}$) *versus* extreme values of contiguity ($C = 0$ and $C = 1$).

The specialized literature (Cooke, 1995) provides the upper (superior limit) and lower (inferior limit) bounds of shear modulus; the numerical results, calculated with these formulas, coincide with those obtained from formula (7) in which the extreme contiguity values have been introduced (Fig. 8)

$$G_{12-} = G_m + \frac{V_f}{\dfrac{1}{G_f - G_m} + \dfrac{V_m}{2G_m}}, \quad G_{12+} = G_f + \frac{V_m}{\dfrac{1}{G_m - G_f} + \dfrac{V_f}{2G_f}}. \quad (9)$$



a



b

Fig. 7 – Variation of shear modulus including the influence of fibres contiguity: *a* – averaging factor $C_{0.785}$; *b* – averaging factor $C_{0.907}$

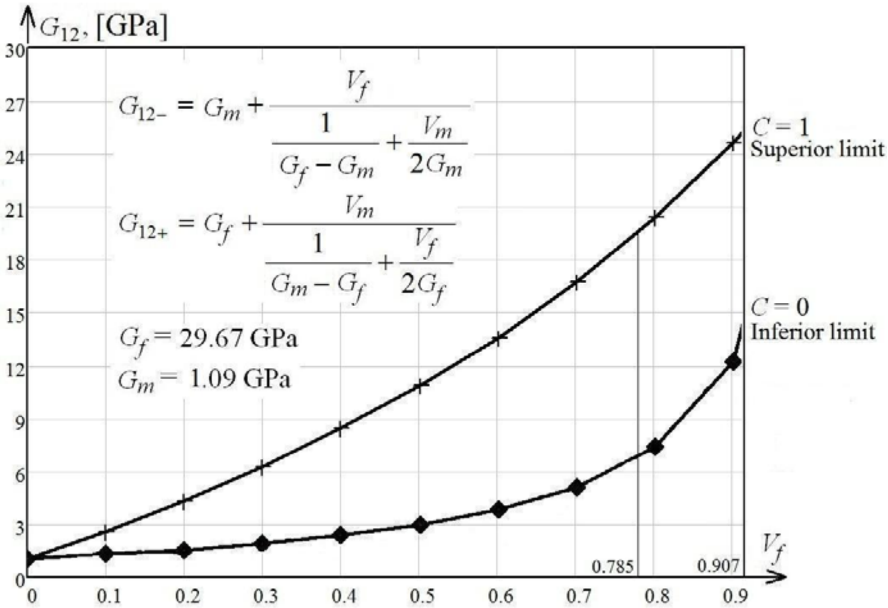


Fig. 8 – Superior and inferior limits for shear modulus.

4.1. Halpin –Tsai Formulas

Halpin and Tsai developed semiempirical eqs. (Halpin, 1992) to match the results of more exact mechanical analyses. These eqs. include some parameters that are influenced by the geometry of the reinforcing fibres, their distribution in the composite and the loading condition,

$$G_{12} = G_m \frac{1+\xi\eta V_f}{1-\eta V_f}, \tag{10}$$

where

$$\eta = \frac{G_f / G_m - 1}{G_f / G_m + \xi}, \tag{11}$$

in which ξ is a factor depending on the fibre geometry, packing geometry and loading condition. A value of $\xi = 1$ has been suggested by Halpin and Tsai for fibres with circular cross section.

Experimental results indicate the adequacy of Halpin-Tsai eqs. to predict the shear modulus for practical requirements. Fig. 9 presents the influence of contiguity, on shear modulus, *versus* extreme values of contiguity (C = 0 and C = 1). As it can be noticed in Fig. 9 the Halpin-Tsai numerical results coincide with those based on the contiguity factor (eq. (7)) when C = 0.

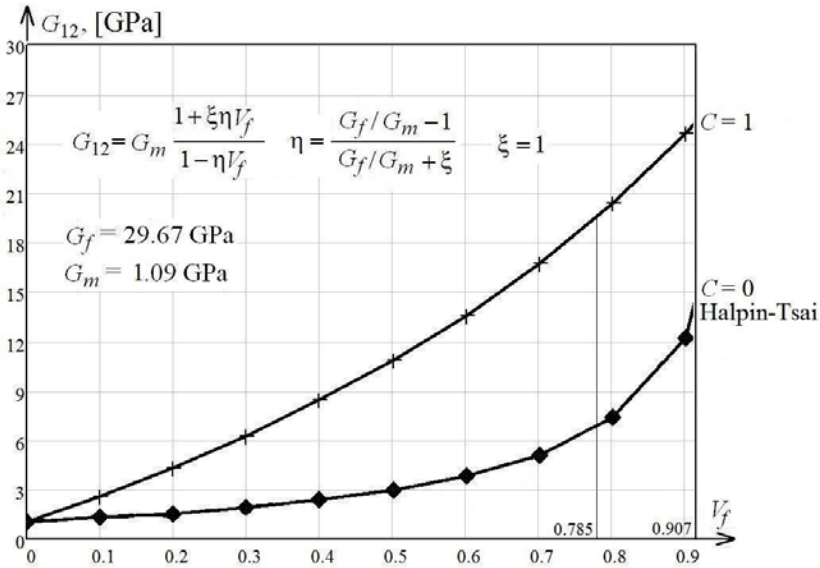


Fig. 9 – Shear modulus predicted through Halpin-Tsai equations *versus* influence of contiguity factor.

4.2. The Composite Cylinder Assemblage (CCA) model

This model enables the exact analytical evaluation of the effective elastic moduli (Jones, 1999). The model consists of an assemblage of composite cylinders (Fig. 10 *a*) each made of a circular fibre core and a concentric matrix shell (Zweben, 1995).

In each cylinder the fibre volume fraction is kept constant (also meaning that the ratio r^2/R^2 is the same); each composite cylinder behaves as an equivalent homogeneous cylinder. The volume of the material is progressively filled out with composite cylinders with different radii. Consequently, the properties of the assemblage approach the properties of one composite cylinder, The following formula corresponds to CCA model:

$$G_{12} = G_m \frac{G_m V_m + G_f (1+V_f)}{G_f V_m + G_m (1+V_f)}. \tag{12}$$

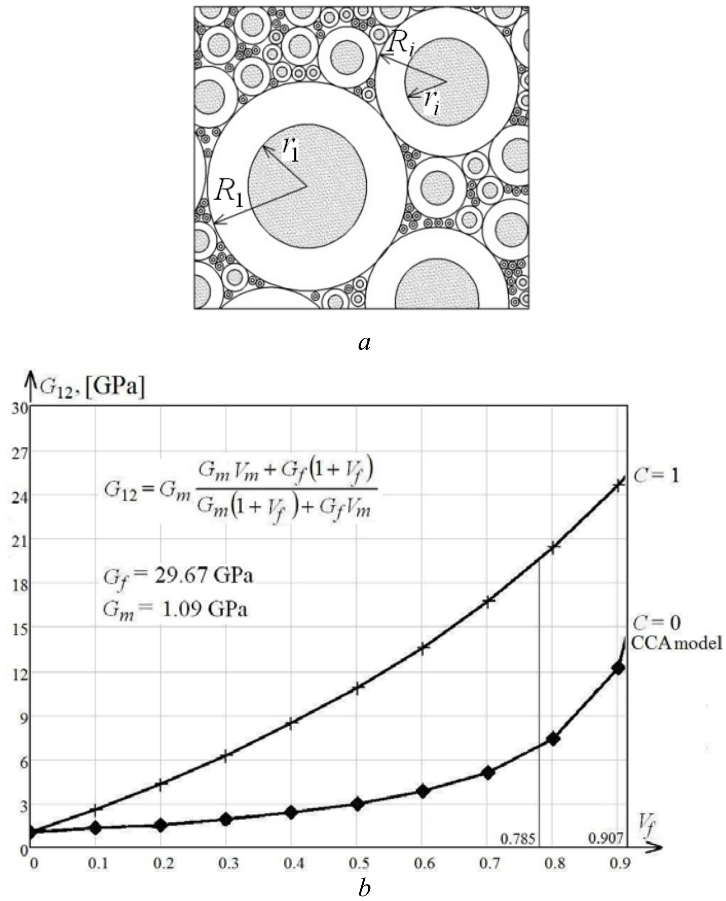
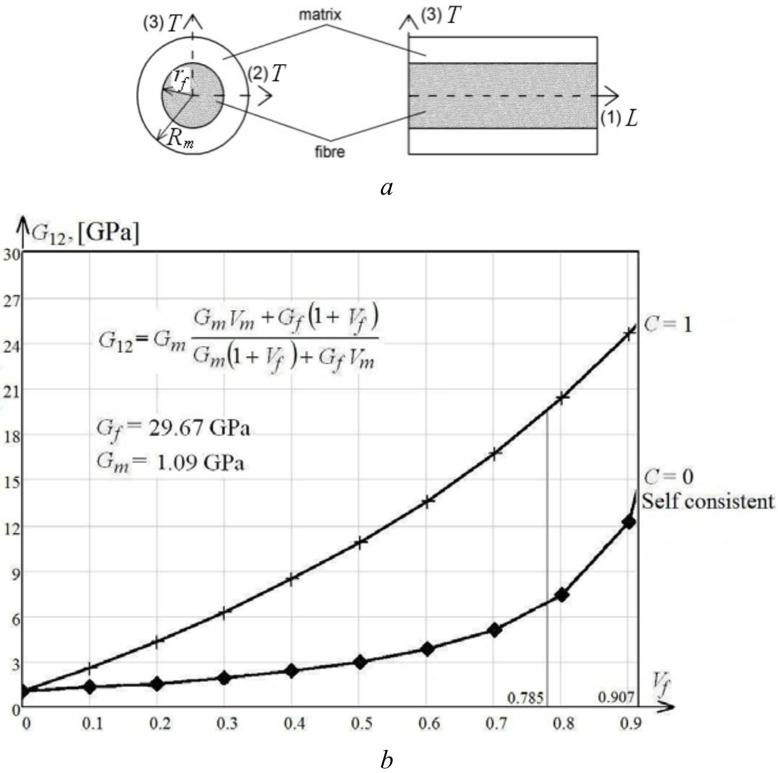


Fig. 10 – CCA model and evaluation of G_{12} including the influence of contiguity: *a* – CCA model; *b* – influence of contiguity.

Fig. 10 *b* presents the comparison of shear modulus values in terms of fibre volume fraction of the CCA model with respect to the extreme values of contiguity.

4.3. The Self-Consistent Model

The model gives one of the most exact solutions and it is described in engineering terms by Whitney and Riley in 1966 (Halpin, 1992, Chamis *et al*., 1968). This model has a single fibre embedded in a concentric cylinder of matrix material (Fig. 11 *a* – Genta, 1982). The volume fraction of the fibre embedded in the composite cylinder is the same as that given by all fibres in the composite material. The formula to compute G_{12} in this model is identical to the one used in the CCA model, eq. (12).



$$G_{12} = G_m \frac{G_m V_m + G_f (1 + V_f)}{G_m (1 + V_f) + G_f V_m}$$

Fig. 11 – Self-consistent model: *a* – the constituents assemblage; *b* – variation of shear modulus with fibre content.

5. Results and Conclusions

A synthesis of all results for the shear modulus is presented in Fig. 12. Since the shear modulus is a matrix dominated property, it can be seen from shear modulus curve traced with the inverse rule of mixtures (Fig. 6) that the fibres have a small contribution to the shear modulus of the unidirectional fibre reinforced polymer composites for low and medium fibre fractions.

A large increase of the shear modulus is obtained for very high fibre volume fractions impossible to be achieved with current fabrication procedures, therefore a different fibres architecture may be suggested when a significant shear modulus increase is required in a specific structural application.

Experimental verification of the results provided by the inverse rule of mixture reveals a significant disagreement with theoretical shear moduli values. This mismatch can be explained by the approximations introduced by the series model (Fig. 5), that does not accurately simulate the behaviour of unidirectional composites under shear loading.

New models have been proposed and applied to match the experimental results with the theoretical ones utilizing more refined micromechanics analyses.

The averaging contiguity factors, $C_{0.785}$ and $C_{0.907}$, lead to intermediate shear modulus values corresponding to various fibre volume fractions distributed in square and triangular arrays.

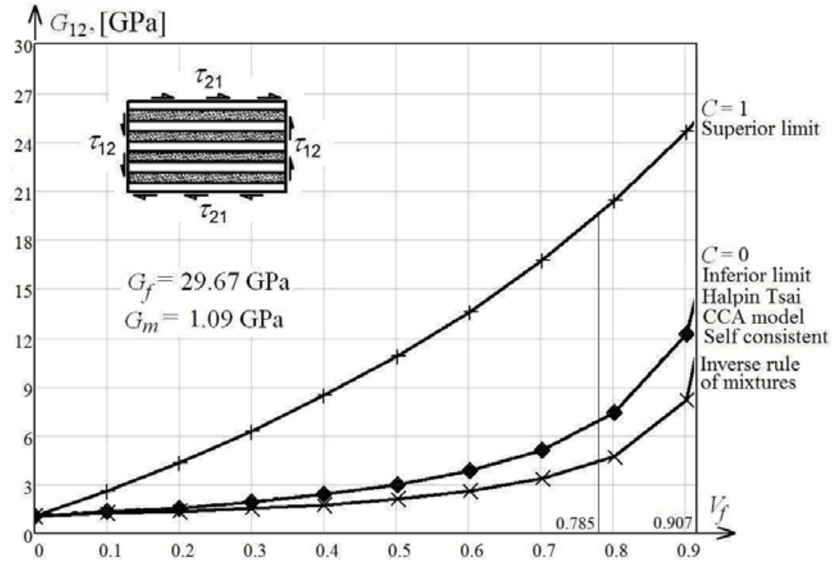


Fig. 12 – Synthesis of all results for the transverse modulus.

As it can be noticed in Fig. 12, the inferior limit of G_{12}, the Halpin-Tsai eqs., the CCA model and the self consistent model give identical values of the composite shear modulus, corresponding to dispersed fibres (C = 0).

REFERENCES

Agarwal B.D., Broutman L.J., Chandrashekhara K., *Analysis and Performance of Fibre Composites*. Third Ed., Wiley-Intersci.,, New-York, 2006.

Askeland D.R., Fulay P.P., *Essentials of Materials Science Engineering*. Second Ed., SI, CENGAGE Learning, Stamford, 2010.

Chamis C.C., Sendeckyj G.P., *Critique on Theories Predicting Thermoelastic Properties of Fibrous Composites*. J. Comp. Mater., **2**, *3*, 332 (1968).

Cooke T.F., *Fibrous Composites: Thermomechanical Properties*. In *Concise Encyclopedia of Composite Materials*. Revised edition, by Kelly A. (Ed.), Pergamon Press, Adv. in Mater. Sci. a. Engng., Oxford, 1995.

Daniel I., Ishai O., *Engineering Mechanics of Composite Materials*. Oxford Univ. Press, Oxford, 2006.

Genta G., *Progettazione Calcolo Strutturale con i Materiali Compositi*. Tecniche Nuove, Milano, 1982.

Gibson R.F., *Principles of Composite Material Mechanics*. Third Ed., CRC Press, Boca Raton, 2012.

Halpin J.C., *Primer on Composite Materials Analysis*. Technomic, Lancaster, 1992.

Hollaway L., *Fibre Composites, in Construction Materials, their Nature and Behaviour*, Part 7, Domone P., Illston J. (Eds.), Spon Press, London, 2010.

Jones R.M., *Mechanics of Composite Materials*. Taylor & Francis Inc., Philadelphia, 1999.

Mallick P.K., *Fiber-Reinforced Composites. Materials, Manufacturing and Design*. Third Ed., CRC Pess, Boca Raton, 2008.

Tsai S.W., Hahn H.T., *Introduction to Composite Materials*. Technomic, Lancaster, 1980.

Țăranu N., Bejan L., *Mecanica mediilor compozite armate cu fibre*. Edit. Cermi, Iaşi, 2005.

Zweben C., *Fibrous Composites: Thermomechanical Properties*. In *Concise Encyclopedia of Composite Materials*. Revised Ed., by Kelly A. (Ed.), Pergamon Press, Adv. in Mater. Sci. a. Engng., Oxford, 1995.

PROPRIETĂȚI MECANICE ALE MATRICELOR MINERALE CU COMPONENȚI ECOLOGICI: DETERMINAREA REZISTENȚELOR LA ÎNCOVOIERE ŞI LA COMPRESIUNE

(Rezumat)

Compozitele polimerice armate cu fibre (CPAF) unidirecționale sunt astăzi utilizate în aplicații variate în ingineria structurală. Multe structuri compozite sunt formate prin stivuire de lamele armate unidirecțional într-o anumită aranjare care să satisfacă criteriile de performanță privind rezistența şi rigiditatea. Ecuațiile constitutive pentru lamelele ortotrope necesită cunoaşterea constantelor elastice asociate axelor principale ale materialului. Modulul de elasticitate la forfecare a unei lamele compozite este dominat de proprietățile matricei şi poate fi determinat analitic utilizând regula inversă a amestecurilor. Totuşi rezultatele astfel obținute au valori mult mai scăzute decât valorile experimentale şi sunt necesare îmbunătățiri ale modelelor pentru o evaluare verosimilă. Lucrarea prezintă cele mai apropiate modele şi, în special, influența factorului de contiguitate, considerând variația fracțiunii volumetrice de fibră pentru diferite aranjări geometrice.