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COMPARATIVE ANALYSIS OF THE BENDING THEORIES FOR ISOTROPIC PLATES. CASE STUDY

BY

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Abstract. This paper presents an overview of the governing eqs. for the bending study of the isotropic plates, with several known plate theories from the literature. For the high order theories (Mindlin and Reddy), which take in consideration the transverse shear strains, are mentioned the differences compared to the classical plate theory (Kirchhoff). Furthermore a case study is presented on a reinforced concrete plate with the purpose to find the displacements relative errors of Mindlin and Kirchhoff plate theories compared to a 3-D analysis. The plate thickness is the parameter which was varied from 0.1 to 0.8 m. There were two softwares used in the comparison: SAP2000 and ANSYS 12, each of these two having both Mindlin and Kirchhoff plate theories implemented. In the end it is shown for both softwares the variation of the relative error for the displacement from the middle of the plate with the thickness of the plate.

Key words: plate theories; governing eqs.; bending displacement, relative error.

1. Introduction

The plates are bidimensional structural elements which have the dimensions from the mid-plane larger than the thickness. Depending of the ratio

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between the minimum dimension from plane ($l_{min} = \min(a,b)$) and the thickness h (Fig. 1), the plates can be classified for the calculation in two classes (Timoshenko & Woinowski-Krieger, 1968; Pank, 1975; Rehfield & Valisettz, 1984; Steek & Balch; Bia *et al.*, 1983):

a) thin plates, at which $l_{min} > 5h$;

b) thick plates, at which $l_{min} < 5h$.

There is no unanimity regarding this classification, some authors considering the ratio $l_{min}/h > 10$ (Wang *et al.*, 2000; Qatu, 2004; Mazilu *et al.*, 1986).

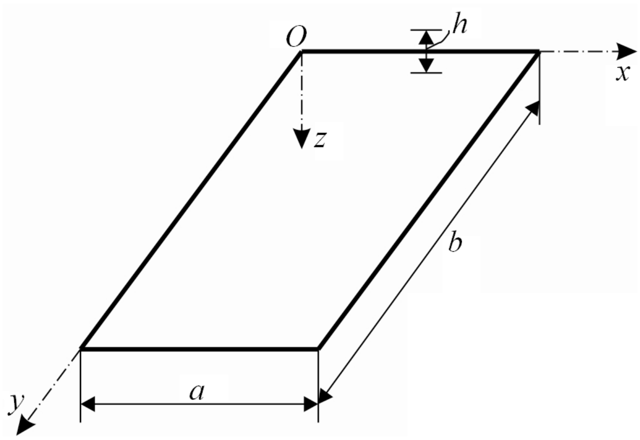

Fig. 1 – Geometric elements and the reference system of the plate.

Due to small thickness, for the thin plates can be used the most frequently 2-D theories, while the thick plate model requires the use of 3-D elasticity theories (Rehfield & Valisettz, 1984; Reissner, 1985; Wang *et al.*, 2000). Governing eqs. of the thin plates can be obtained using vectorial mechanics or variational and energetic principles. In vectorial mechanics the internal forces and the bending moments which are disposed on a typical plate differential element are summed with the purpose to obtain the equilibrium or motion eqs. In the energetic methods in order to obtain the eqs., various types of virtual work principles are used, such as the principles of minimum potential energy or the complementary potential energy (Wang *et al.*, 2000; Atanackovic & Guran, 2000; Vrabie & Ungureanu, 2012).

The bidimensional plate theories can be classified in (Pank, 1975; Rehfield & Valisettz, 1984; Wang *et al.*, 2000):

1° Classical plate theory, in which the effects of transverse shear strain are neglected.

2° Plate theories, in which the shear strains are considered.

Further, an overview of these theories is done highlighting the differences and also a case study is done in two different softwares with the purpose to find the displacements relative errors of Mindlin and Kirchhoff plate theories compared to a 3-D analysis.

2. The Displacement Field

The plate theories developed into the literature are based on the adoption of a form of the displacement field like a linear combination of unknown functions and on coordinate on thickness direction

$$\varphi_i\left(x,y,z,t\right)=\sum_{j=1}^{n}(z)^i\varphi_i^j\left(x,y,z\right), \qquad (1)$$

where: φ_i is the "i" component of displacement, (x,y) are the coordinates from the mid-plane of the plate, z is the thickness coordinate, t is the time and φ_i^j are functions which must be determinated.

The classical bending plate theory (Kirchhoff) is based on the following displacement field (Timoshenko & Woinowski-Krieger, 1968; Pank, 1975; Vrabie & Ungureanu, 2012; Steek & Balch; Wikipedia):

$$u\left(x,y,z\right)=-z\frac{\partial w_0}{\partial x},\;\; v\left(x,y,z\right)=-z\frac{\partial w_0}{\partial y},\;\; w\left(x,y,z\right)=w_0\left(x,y\right), \qquad (2)$$

where: (u, v, w) are displacement components of a point along the (x, y, z) co-ordinate directions, respectively, and w_0 is the transverse deflection of a point from the mid-plane (*i.e.*, $z = 0$).

The adopted displacement field implies the fact that a normal rectilinear segment on the mid-plane before deformation remains straight and normal on middle surface after the deformation of the plate (Kirchhoff hypothesis). This hypothesis permits neglect the transverse shear effects ($\tau_{xz}= \tau_{yz}= 0$), but also the normal ones ($\sigma_z = 0$). In other words the plate deformations are given entirely by bending and the axial forces (Fig. 2 *a*).

The simplest plate theory with shear deformation is the *first order* one (Mindlin - Reissner), which is based on the following displacement field (Reissner, 1985; Wang *et al*., 2000; Qatu, 2004):

$$u\left(x,y,z\right)=-z\phi_x\left(x,y\right),\;\; v\left(x,y,z\right)=-z\phi_y\left(x,y\right),\;\; w\left(x,y,z\right)=w_0\left(x,y\right), \qquad (3)$$

where: ϕ_x and ϕ_y are the rotations reported at x and y axis (Fig. 2 *b*).

The Mindlin- Reissner theory includes in the eqs. (3) a global transverse shear strain considered constant on the plate thickness. To correct the discrepancy between the real distribution of the transverse shear force and the

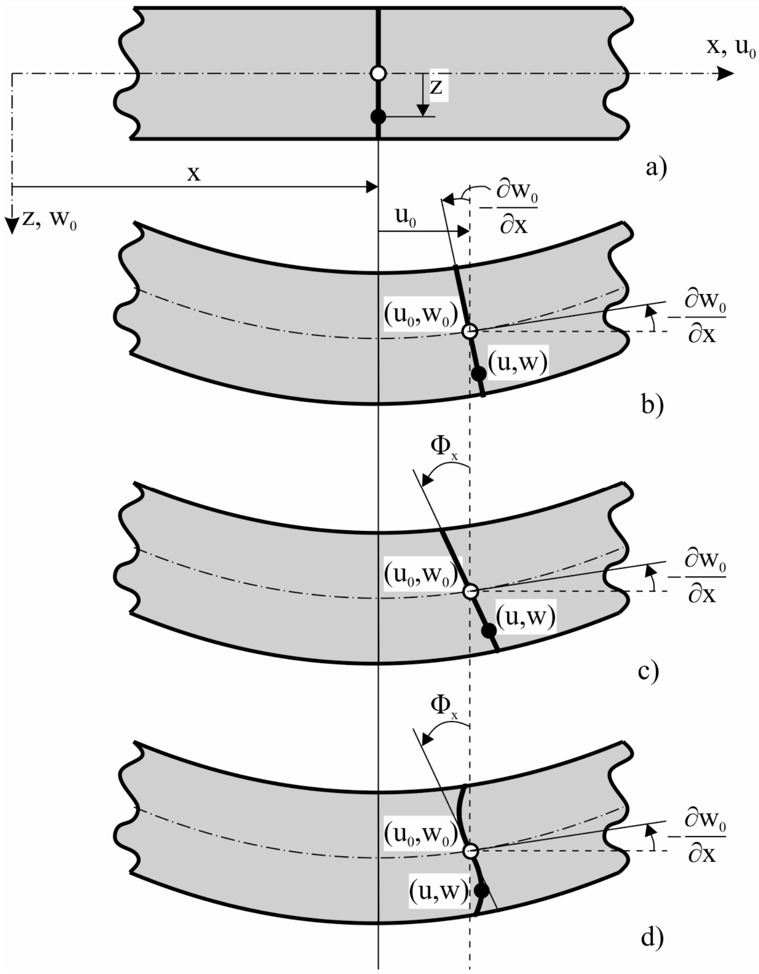

Fig. 2 – Undeformed and deformed geometries of an edge before and after deformation in the studied plate theories: *a* – undeformed; *b* – deformation in classical theory (Kirchhoff); *c* – deformation in Mindlin theory; *d* – deformation in Reddy theory.

one resulted from the utilization of the kinematic relations is introduced a shear correction factor. This factor depends on both the geometric parameters as the plate loading and restraints. The second order and the high order theories with shear deformations use high order polynomial functions to extend displacement components in the *z*-axis direction. The higher order theories introduce additional unknowns which often is difficult to interpret them physically. An example is represented by the second order theory with transversal inextensibility which is based on the following displacement field:

$$\begin{aligned} u(x,y,z) &= z\phi_x(x,y) + z^2\psi_x(x,y), \\ v(x,y,z) &= z\phi_y(x,y) + z^2\psi_y(x,y), \\ w(x,y,z) &= w_0(x,y), \end{aligned} \tag{4}$$

where: ψ_x and ψ_y are unknown functions without physically signification.

From the *third order theories*, the most known is the one of Reddy (Wang *et al*., 2000), which proposed the following displacement field:

$$\begin{aligned} u(x,y,z) &= z\phi_x(x,y) - \alpha z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right), \\ v(x,y,z) &= z\phi_y(x,y) - \alpha z^3\left(\phi_y + \frac{\partial w_0}{\partial y}\right), \\ w(x,y,z) &= w_0(x,y), \end{aligned} \tag{5}$$

where: $\alpha = 3/4h^2$ (Fig. 2 *c*).

The displacement field leads to a quadratic variation of transverse shear strain (and of corresponding stresses that are null on the superior and inferior faces of the plate). It must be noted that the third order theory doesn't need a shear correction factor, and for $\alpha = 0$ we can obtain the displacement field from the first order theory.

3. Comparisons and Linking Relations Between Kirchhoff and Mindlin Theories

In order to avoid the confusions between various response parameters in these plate theories, the parameters will be indexed above with *K*, in the classical theory (Kirchhoff) and with *M*, in the first-order shear deformation plate theory (Mindlin; Wang *et al*., 2000).

Using the mentioned notations, the biharmonic eq. $D\nabla^4 w(x, y) = q(x, y)$, governing plate bending according to the Kirchhoff theory, can be expressed as a pair of Poisson eqs.

$$\nabla^2 M^K = -q \quad \text{a)}; \qquad \nabla^2 w^K = -\frac{M^K}{D}, \quad \text{b)} \tag{6}$$

where M^K is the sum moment (or Marcus moment)

$$M^K = \frac{M_x^K + M_y^K}{1+v} = -D\left(\frac{\partial^2 w^K}{\partial x^2} + \frac{\partial^2 w^K}{\partial y^2}\right) = -D\nabla^2 w^K. \tag{7}$$

Similarly, the governing eqs. of static equilibrium of plates according to the Mindlin plate theory, can be expressed in terms of the deflection, w^M , and the moment sum, M^M

$$\nabla^2 M^M = -q \quad \text{a)}; \qquad \nabla^2\left(w^M - \frac{M^M}{k_s Gh}\right) = -\frac{M^K}{D} \quad \text{b)};$$
$$M^M = \frac{M_x^M + M_y^M}{1+v} = D\left(\frac{\partial \phi_x}{\partial x} - \frac{\partial \phi_y}{\partial y}\right), \tag{8}$$

where k_s is parameter, named *shear correction factor*.

After some processing we obtain a more compact form

$$D\nabla^4 w = \left(1 - \frac{D}{k_s Gh}\nabla^2\right)q. \tag{9}$$

Comparing the relations (6 *a*) and (8 *a*) and considering the sum moments expressions M^M and M^K, between them it can be written the following linking relation:

$$M^M = M^K + D\nabla^2\Phi, \tag{10}$$

where Φ is a biharmonic function (such as w^K, which figuress in the eq. of M^K), so that the condition: $\nabla^2\nabla^2\Phi = 0$ be satisfied.

Similarly, comparing the eqs. (6 *b*) and (8 *b*), and considering the eq. (10), after some processing it can be obtained

$$w^M = w^K + \frac{M^K}{k_s Gh} + \frac{D}{k_s Gh}\nabla^2\Phi - \Phi = w^K + \frac{M^K}{k_s Gh} + \Psi - \Phi, \tag{11}$$

where $\Psi = \dfrac{D}{k_s Gh}\nabla^2\Phi$ is a harmonic function ($\nabla^2\Psi = 0$).

If we replace M^K from (8 *b*) and will write D and G in function of E and v, the relation (11) can be written

$$w^M = w^K - \frac{h^2}{6k_s(1-v)}\nabla^2 w^K + \Psi - \Phi. \tag{12}$$

The relation (11) (respectively (12)) is valid for plates with arbitrary loading and restraints. The functions Φ and Ψ can be determined from the restraint conditions on the plate boundary. Limiting the analysis to the case in which $w^M = w^K = 0$ on the border, and M^K is either zero or a constant value M_c^K, the difference $\Psi - \Phi$ is equal to $-M^K/k_sGh$ and the relation (11) can be written:

$$w^M = w^K + \frac{M^K - M_c^K}{k_sGh}. \tag{13}$$

Starting from this linking relation between the displacements in the two plate theories, the other response parameters can be expressed similarly

$$\begin{gathered}
\frac{\partial w^M}{\partial x} = \frac{\partial w^K}{\partial x} + \frac{V_x^K}{k_sGh}; \quad \frac{\partial w^M}{\partial y} = \frac{\partial w^K}{\partial y} + \frac{V_y^K}{k_sGh}; \\
M_x^M = M_x^K + \frac{D(1-\nu)}{k_sGh} \cdot \frac{\partial}{\partial x}\left(V_x^M - V_x^K\right); \\
M_y^M = M_y^K + \frac{D(1-\nu)}{k_sGh} \cdot \frac{\partial}{\partial y}\left(V_y^M - V_y^K\right); \\
M_{xy}^M = M_{xy}^K + \frac{D(1-\nu)}{k_sGh}\left[\frac{\partial}{\partial y}\left(V_x^M - V_x^K\right) - \frac{\partial}{\partial x}\left(V_y^M - V_y^K\right)\right]; \\
V_x^M = V_x^K + \frac{D(1-\nu)}{2k_sGh}\nabla^2\left(V_x^M - V_x^K\right); \\
V_y^M = V_y^K + \frac{D(1-\nu)}{2k_sGh}\nabla^2\left(V_y^M - V_y^K\right).
\end{gathered} \tag{14}$$

In the case of simply restraint polygonal plates (with straight borders), in the Kirchhoff plate theory, the boundary conditions are

$$w^K = M^K = 0. \tag{15}$$

In the Mindlin plate theory, the simply restraint of the border can considered to be "soft" or "hard". In the last case we can obtain similar conditions with the classical theory

$$w^M = M^M = 0. \tag{16}$$

Because on these borders we have $M^M = 0$, it results also $M_c^M = 0$, from these the relation (13), so that the displacement becomes

$$w^M = w^K + \frac{M^K}{k_sGh}. \tag{17}$$

The relation (17) links the bending displacement of the plate in Mindlin theory by the w^K and M^K of the plates from the classical theory (Wang *et al.*, 2000).

4. The Case Study

The case of study considers a reinforced concrete plate. The plate has a square form in plan with the length of the edge of 6 m. In order to do a parametric analysis and to compare the results between Kirchhoff and Mindlin plate theories the thickness was varied from 0.1 to 0.8 m. In the analyses two restraint situations were consideres, with fully restrained edges (fixed or clamped edges) and with simply supported edges. Moreover the analyses were done in two different softwares: ANSYS 12 (Fig. 3) (ANSYS 12, 2009) and SAP2000 (Fig .4) (Sap 2000, 2009), in order to see which give better results. The purpose of this case study was to find the relative errors of the displacements from the middle of the plate between the complex 3-D theory and the 2-D theories, namely: classical theory (Kirchhoff) and the first order shear deformation theory (Mindlin). Then the variations of these relative errors with the plate thickness are plotted. The 3-D theory is considered to be more accurate but also time consuming due to large number of degree of freedom. To overcome this drawback the plate theories have been implemented in these softwares.

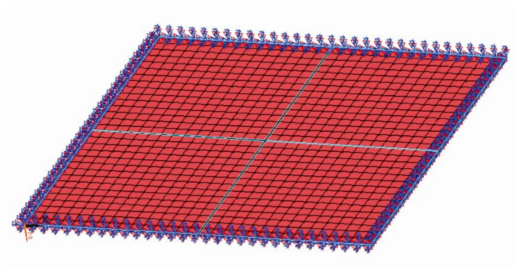

Fig. 3 – Plate modeled in ANSYS 12 software.

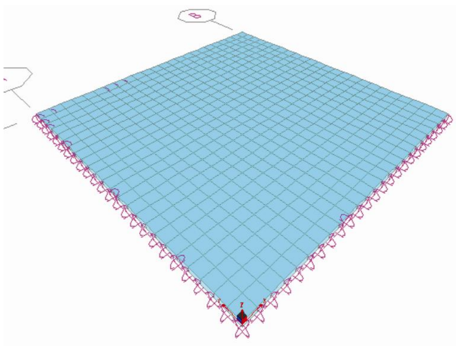

Fig. 4 – Plate modeled in SAP2000 software.

4.1. Analysis of the Fully Restrained (Clamped) Plate

Firstly, the plate was modeled in ANSYS software in three different versions:

a) With a 3-D model using SOLID65 finite element.

b) With a 2-D model using SHELL181 finite element (Mindlin theory).

c) With a 2-D model using SHELL63 finite element (Kirchhoff theory) (ANSYS 12, 2009).

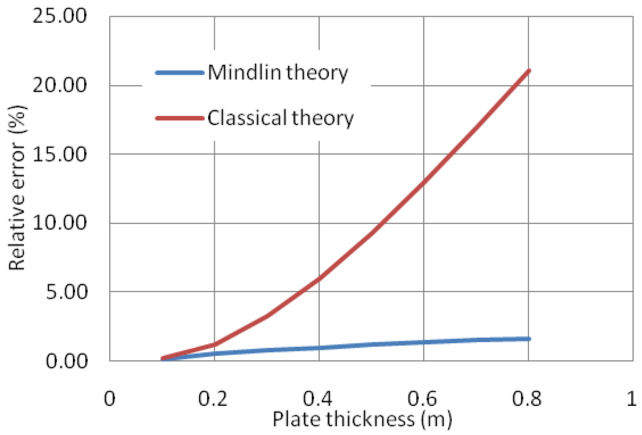

Fig. 5 – The variation of the displacement relative error with the plate thickness for the fully restrained plate in ANSYS 12 software for Classical and Mindlin theories.

The line chart from Fig. 5 shows the variation of the relative error of the displacement from the middle of the plate with the plate thickness for both Kirchhoff and Mindlin theories. The relative errors were computed for both theories in comparison with the 3-D theory. It can be noticed that for a thin plate of 0.1 m the relative errors are very low, close to 0. It would appear that increasing the thickness of the plate the errors increase too. Therefore the error in Mindlin theory has a slight increase and reaches to a maximum of 1.6% while the error of the classical theory (Kirchhoff) has a sharp increase reaching to a maximum value of 21% for the plate thickness of 0.8 m, resulting that the total difference between those two theories is almost 20%.

In SAP2000 software, there are the following options implemented in order to simulate plates: the *thin plate model* and the *thick plate model*. The thin plate option uses the classical theory (Kirchhoff) while the thick plate option uses the Mindlin theory (SAP 2000, 2009). In this case, the relative error is computed also in comparison with the 3-D model from ANSYS12, because this software has only 2-D models implemented. As we can notice the results are very close to those obtained in ANSYS12 software (Fig.6). Therefore it would appear that in both softwares the displacements obtained with Mindlin theory give small errors and are very accurate no mather how thick the plate is. On the other hand, the relative error of displacements obtained with the classical theory are increasing with the thickness of the plate. For example the relative error of the displacement in the middle of the plate for a thickness of 0.1 and 0.2 m is approximately 1%, for a thickness of 0.5 m is 10% and reaching at 21% for a 0.8 m thickness.

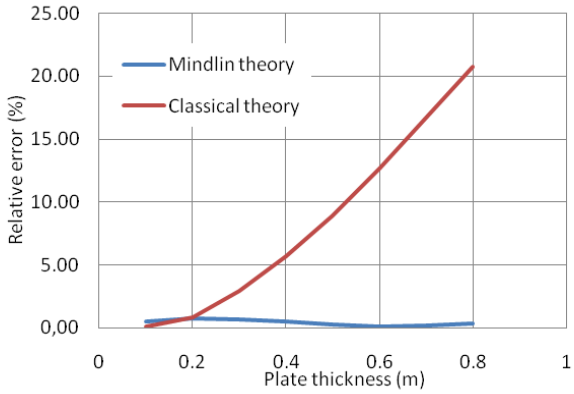

Fig. 6 – The variation of the displacement relative error with the plate thickness for the fully restrained plate in SAP2000 software for Classical and Mindlin theories.

4.2. Analysis of the Simply Supported Plate

The same analyses like the previous ones were done for a simply supported plate, with the same dimensiones and properties. For these analyses the plate was computed also with classical and Mindlin theories by FEM in ANSYS 12 and SAP2000 software but also with Fourier series. The Fourier series analytical solutions are close to the ones obtained in the classical theory.

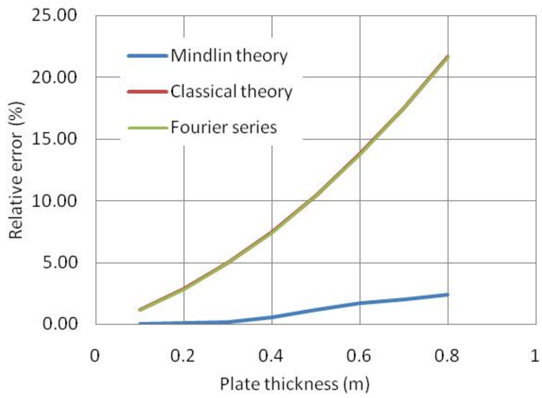

Fig. 7 – The variation of the displacement relative error with the plate thickness for the simply supported plate in ANSYS 12 software for Classical, Mindlin and Fourier series theories.

As it can be seen in Fig. 7 the results obtained with ANSYS 12 software are close to those from fully restrained plate. On the other hand the results obtained in SAP2000 software give higher errors for the Mindlin theory (Fig.8). This may be due to inexact simulation of the plate restraints. For the shell models the edges are simply supported on the nodes from the middle plan while for the 3-D model the simply support is simulated at the bottom nodes of the edge.

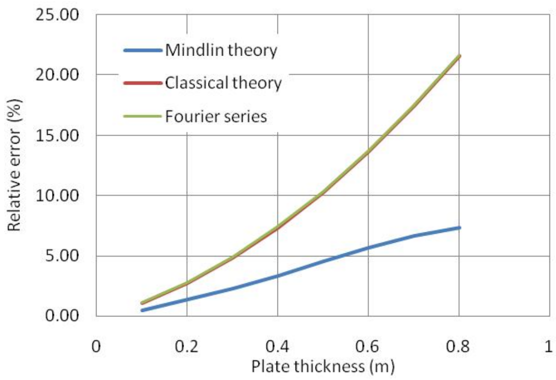

Fig. 8 – The variation of the displacement relative error with the plate thickness for the simply supported plate in SAP2000 software for Classical, Mindlin and Fourier series theories.

5. Conclusions

The calculation of the plate bending response parameters is done often using the classical theory. The simplicity of this theory is based on the normal rectilinear and inextensible segment hypothesis (Kirchhoff hypothesis). This hypothesis neglects the transverse shear effects (strains, stresses). As a result the bending displacement calculated in classical theory is underestimated. The high order theories take in consideration the transverse shear effects through kinematic field relaxation, this being reflected in the selection of the displacement field.

The first order shear deformation plate theory (Mindlin) relaxes the normal segment hypothesis and takes into consideration a constant shear strain on the plate thickness. In order to correct the discrepancy between this constant distribution and the real parabolic distribution a shear correction factor, k_s, was introduced.

The third order shear deformation plate theory (Reddy) relaxes more the cinematic hypotheses by adopting a displacement field with a cubic variation on the thickness plate for both u and v displacements. This high order theory can be avoided because the complexity of calculation is high and the surplus of precision can be neglected. The researchers' unanimous conclusion is that Mindlin theory is sufficient for the required engineering accuracy. However, the Mindlin theory has also enough complications compared to the classical theory. That why, through the analysis and comparison between the governing eqs. of the two theories, it were established some relations which link the plate response parameters from the Mindlin theory with the ones from the classical theory. In this way, for plates with polygonal perimeter, as for the axial symmetric circular plates, simply supported on border, it was establish the relation (17), which permits the calculation of the Mindlin plate displacement with the displacement and the sum moment from the Kirchhoff theory. The bending response parameters, w^k and M^k, are easier to calculate and, often, can be found in the literature, avoiding in this way the plate bending analyses with shear strain. As a consequence of that, the finite elements having the Mindlin theory implemented show a softer deformation behavior due to the presence of shear stresses. According to some autors (Banarjee *et al*., 2011), the effects of shear deformation can be neglected as long as the shell ratio h/L is less than 1/10, where h is the thickness and L is the edge length. For both shell theories the bending stresses vary linearly with respect to the thickness.

It is very important that the users to choose the correct finite shell element in order to obtain good results. The recommendation is that in ANSYS12 software the users should take the SHELL181 finite element while in SAP2000 software should use the thick plate option for accurate results. In spite of the fact that some authors consider that the shear deformations can be neglected for a ratio less than 1/10, these analyses revealed that at this ratio the relative error reaches approximately to 15%.

REFERENCES

Atanackovic T.M., Guran A., *Theory of Elasticity for Scientists and Engineers.* Birkhauser Boston, 2000.

Banarjee B., Chen J., Das R., Kathirgamanathan A., *A Comparison of ANSYS Elements SHELL181 and SOLSH190.* Res. Rep., Univ. of Auckland, NewZeeland, 2011.

Bia C., Ille V., Soare M., *Rezistența materialelor şi teoria elasticității*. Ed. Did. şi Pedag., Bucureşti, 1983.

Mazilu P., Țopa N., Ieremia, M., *Aplicarea teoriei elasticității şi a plăcilor în calculul construcțiilor*. Ed. Tehnică, Bucureşti, 1986.

Pank Vl., *Theories of Elastic Plates.* Noodhoff Internat. Publ., Leyden, 1975.

Qatu M., *Vibration of Laminated Shells and Plates*. Academic Press, 2004.

Rehfield L.W., Valisettz R.R., *A Simple, Refined Theory for Bending and Stretching of Homogeneous Plates.* AIAA J., **22**, *1*, 90-95 (1984).

Reissner E., *Reflections on the Theory of Elastic Plates*. Appl. Mech. Rev., **38**, *11*, 1453-1464 (1985).

Steek C., Balch C., *Introduction to the Theory of Plates*. http://www.stanford.edu/chasst/course%20Notes/PlatesBook.

Timoshenko S., Woinowski-Krieger S., *Teoria plăcilor plane şi curbe* (transl. From English). Ed. Tehnică, Bucureşti, 1968.

Vrabie M., Ungureanu N., *Calculul plăcilor – teorie şi aplicații*. Ed. Soc. Academice "Matei-Teiu Botez", Iaşi, 2012.

Wang C.M., Reddy J.N., Lee K.H., *Shear Deformable Beams and Plates*. Elsevier Science Ltd., 2000.

* * * ANSYS 12 Structural Analysis Guide, 2009.

* * * *Plate Theory*. Wikipedia.

* * * *Sap 2000 V14 Help*. Computers and Structures Inc., Berkeley, California, USA, 2009.

ANALIZA COMPARATIVĂ A TEORIILOR DE ÎNCOVOIERE PENTRU PLĂCILE PLANE IZOTROPE. STUDIU DE CAZ

(Rezumat)

Se prezintă o trecere în revistă a ecuațiilor de guvernare pentru studiul la încovoiere a plăcilor plane izotrope, fiind luate în considerare câteva teorii cunoscute din literatura de specialitate. Pentru teoriile de ordin superior (Mindlin şi Reddy), care iau în considerare efectul deformațiilor transversal, sunt menționate diferențele în comparație cu teoria clasică a plăcilor plane (Kirchhoff). În continuare, este prezentat un studiu de caz asupra unei plăci plane din beton armat, cu scopul de a pune în evidență erorile relative la calculul deplasărilor obținute prin teoriile 2-D, Mindlin şi Kirchhoff, în comparație cu teoria 3-D. Grosimea plăcii este variată în aceasta analiză de la 0.1 m până la 0.8 m. Au fost folosite două programe de calcul, SAP2000 şi ANSYS12, ambele având integrate teoriile plăcilor plane Mindlin şi Kirchhoff. În final sunt trasate variațiile erorilor relative obținute la calculul deplasării din centrul plăcii relativ la grosimea acesteia, pentru ambele programe de calcul folosite.