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ABOUT THE FINITE ELEMENT METHOD APPLIED TO THICK PLATES

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The present paper approaches the problem of plates subjected to transverse loads, when the shear force and the actual boundary conditions are considered, by using the Finite Element Method.

The isoparametric finite elements create real facilities in formulating the problems and great possibilities in creating adequate computer programs.

1. Introduction

The design of thick plates acted by transverse loads, using the thin plate's theory, does not correspond to theoretical and practical requirements, but a single theory could include both categories: thin and thick plates. A great number of research works about the design of thick plates exists. The most important initiators of this problem are R e i s s n e r and M i n d l i n [3],...,[5]. Each of them proposed a special model, but they present several difficulties in approaching the plate problems.

In the thin plates theory some important aspects should be taken into account: the shear force effect, the real boundary conditions, the effect of concentrated forces, the effect of thickness variation, the effect of holes and other types of stress concentrators.

The assumption of neglegting the deformations along the plate thickness is maintained, therefore, the axial strain $\varepsilon_z = 0$, z being the axis of the coordinate system, normal to plate middle plane. The other two axes, x and y, are obviously the orthogonal axes in the plate plane.

The pressure between the plate fibers, that is normal stress σ_z, is considered equal to zero, too.

The assumption of normal line element (Kirchhoff), which is generally used in the shell theory, is substituted by Mindlin assumption. According to it, a line element normal to the middle plane of the undeformed plate remains straight, but not necessarily normal to the middle plane of the deformed plate. The consequences of this assumption are the following:

a) The angles between the normal lines to the deformed middle plane and those to the undeformed middle plane are φ_x and φ_y, in xOz plane and yOz plane, respectively, different from the derivatives $\partial w/\partial x$ and $\partial w/\partial y$. Therefore, the three displacements in a point, w, φ_x, φ_y, are relatively independent ones.

b) The linear displacements, u and v, in a point located at the distance z from the middle plane, become

$$u = z\varphi_x, \qquad v = z\varphi_y. \tag{1}$$

The displacements w, φ_x and φ_y depend on coordinates x, y, while u and v depend linearly on z, too.

The deformations are further expressed by using Cauchy equations and then, the stresses are derived, according to Hooke's law.

The stress (τ_{xz} and τ_{yz}) produced by the transverse loads (the shear force effect) lead to the section warping, because they are non-uniformly distributed over the plate thickness.

2. Strains. Stress. Internal Forces

The strains are expressed in terms of displacements by using the geometrical equations (Cauchy equations)

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} = z\dfrac{\partial \varphi_x}{\partial x}, \quad \varepsilon_y = \dfrac{\partial v}{\partial y} = z\dfrac{\partial \varphi_y}{\partial y}; \\ \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = z\dfrac{\partial \varphi_x}{\partial y} + z\dfrac{\partial \varphi_y}{\partial x}, \quad \gamma_{zx} = \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} = \dfrac{\partial w}{\partial x} + \varphi_x, \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} = \dfrac{\partial w}{\partial y} + \varphi_y. \end{cases} \tag{2}$$

The constitutive law of the material (Hooke's law) leads to the following stresses:

$$\begin{cases} \sigma_x = \dfrac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = \dfrac{Ez}{1-\nu^2}\left(\dfrac{\partial \varphi_x}{\partial x} + \nu\dfrac{\partial \varphi_y}{\partial y}\right), \\ \sigma_y = \dfrac{E}{1-\nu^2}(\nu\varepsilon_x + \varepsilon_y) = \dfrac{Ez}{1-\nu^2}\left(\nu\dfrac{\partial \varphi_x}{\partial x} + \dfrac{\partial \varphi_y}{\partial y}\right), \\ \tau_{xy} = \dfrac{E}{2(1+\nu)}\gamma_{xy} = \dfrac{Ez}{2(1+\nu)}\left(\dfrac{\partial \varphi_x}{\partial y} + \dfrac{\partial \varphi_y}{\partial x}\right). \end{cases} \tag{3}$$

The internal forces and moments are obtained using the equivalence relations

$$(M_x, M_y, M_{xy}) = \int\limits_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) z \,\mathrm{d}z, \quad (Q_x, Q_y) = \int\limits_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) \,\mathrm{d}z, \tag{4}$$

where h is the plate thickness.

By introducing the stresses from (3) in (4) and integrating, it results

$$(5)\qquad \begin{cases} M_x = D_M\left(\dfrac{\partial\varphi_x}{\partial x} + \nu\dfrac{\partial\varphi_y}{\partial y}\right), \quad Q_x = D_Q\left(\dfrac{\partial w}{\partial x} + \varphi_x\right), \\ M_y = D_M\left(\nu\dfrac{\partial\varphi_x}{\partial x} + \dfrac{\partial\varphi_y}{\partial y}\right), \quad Q_y = D_Q\left(\dfrac{\partial w}{\partial y} + \varphi_y\right), \\ M_{xy} = \dfrac{1-\nu}{2}D_M\left(\dfrac{\partial\varphi_x}{\partial y} + \dfrac{\partial\varphi_y}{\partial x}\right), \end{cases}$$

where

$$(6)\qquad D_M = \frac{Eh^3}{12(1-\nu^2)}; \quad D_Q = \frac{Ehk_Q}{2(1+\nu)}.$$

k_Q is a factor that takes into account the non-uniform distribution of shear stresses, τ_{xz} and τ_{yz}, over the plate thickness.

The previous expressions could be extended to anisotropic plates, as well as to orthotropic ones.

3. Isoparametric Finite Elements

There are considered quadrilateral finite element with four and eight nodes, defined in natural coordinates, ζ and η (Fig. 1).

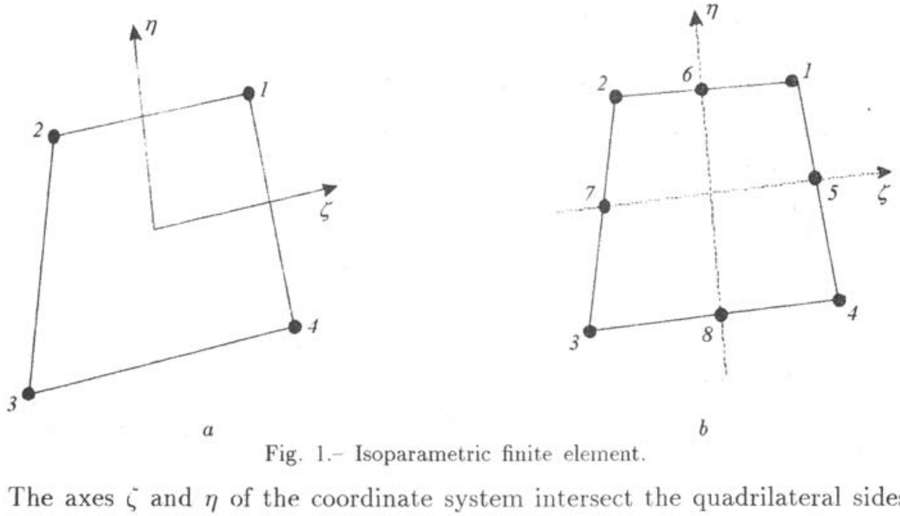

Fig. 1.– Isoparametric finite element.

The axes ζ and η of the coordinate system intersect the quadrilateral sides at their midpoints.

The system $(O\,\zeta\,\eta)$ is a dimensionless one. The coordinates of the nodes for the quadrilateral finite element with four nodes (Fig. 1 *a*) are: *1*(1, 1); *2*(–1, 1); *3*(–1, –1); *4* (1, –1).

For the finite element with eight nodes (Fig. 1 *b*), the coordinates of the corner nodes are the same, and the coordinates of the nodes from the side midpoints are: *5*(1/2, 0); *6*(0, 1/2); *7* (−1/2, 0); *8*(0, −1/2).

For each node, i, three degrees of freedom (DOF) are considered: the deflection, w_i, normal to the middle plane and two slopes, φ_{x_i} in xOz plane and φ_{y_i} in yOz plane.

In order to determine the coordinates (x, y) at a point located on the finite element surface, the nodes coordinates (x_i, y_i) and the shape functions, $N_i(\zeta, \eta)$, are used.

Similarly, in order to determine the displacements at a point on the finite element surface, $(w, \varphi_x, \varphi_y)$, the nodal displacements $(w_i, \varphi_{x_i}, \varphi_{y_i})$, $(i = \overline{1, n}$; n – the number of nodes) and the same shape functions $N_i(\zeta, \eta)$ are needed.

The coordinates (x, y) have the following expressions:

$$x = \sum_{i=1}^{m} N_i(\zeta, \eta) x_i \qquad y = \sum_{i=1}^{m} N_i(\zeta, \eta) y_i, \tag{7}$$

where m is the number of nodes associated to the finite element. In our cases $m = 4$ and $m = 8$, respectively.

The displacement field $(w, \varphi_x, \varphi_y)$ is defined in a similar manner

$$w = \sum_{i=1}^{m} N_i(\zeta, \eta) w_i \qquad \varphi_x = \sum_{i=1}^{m} N_i(\zeta, \eta) \varphi_{x_i} \qquad \varphi_y = \sum_{i=1}^{m} N_i(\zeta, \eta) \varphi_{y_i}. \tag{8}$$

The finite element model, which uses the same shape functions for coordinates (x, y) and displacements, $(w, \varphi_x, \varphi_y)$, is called *isoparametric finite element.*

3.1. Finite Element with Four DOF

The quadrilateral finite element with the nodes located at its corners is the bilinear finite element.

The shape functions, $N_i(\zeta, \eta)$, associated to these four nodes could be obtained by interpolation, and have the form

$$N_i = \frac{1}{4}(1 + \zeta_i \zeta)(1 + \eta_i \eta). \tag{9}$$

For the nodes of the finite element shown in Fig. 2 *a*, the coordinates (ζ_i, η_i), with the corresponding values: *1*(1, 1); *2*(−1, 1); *3*(−1, −1) and *4*(1, −1) are substituted in relations (9) and the shape function N_i, $(i = 1, 2, 3, 4)$, are determined.

3.2. Finite Element with Eight DOF

The nodes of the finite element are provided at its corners (*1,2,3,4*) and at the side midpoints (*5,6,7,8*).

Functions of Serendip type are used

a) nodes at the corners

$$N_i = \frac{1}{4}(1+\zeta_i\zeta)(1+\eta_i\eta)(\zeta_i\zeta+\eta_i\eta-1); \tag{10 a}$$

b) nodes at side midpoints

$$N_i = \frac{1}{2}\zeta_i^2(1+\zeta_i\zeta)(1-\eta^2)+\frac{1}{2}\eta_i^2(1-\eta_i\eta)(1-\zeta^2). \tag{10 b}$$

The finite element with four DOF has straight sides, while the sides of the element with eight DOF could be curved ones.

4. Finite Element Analysis

The carthesian coordinates (x, y) of the points belonging to the finite element surface are determined according to relation (7), which has a vector form

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \sum_i N_i(\zeta,\eta)\begin{Bmatrix} x_i \\ y_i \end{Bmatrix}. \tag{11}$$

Similarly, the displacements at a point of the finite element, w, φ_x, φ_y, are obtained by (12).

The column vector of nodal displacements at a node of the finite elements is

$$\{d_{n_i}\} = \{w_i, \varphi_{x_i}, \varphi_{y_i}\}^T. \tag{12}$$

The column vector of nodal displacements for the whole finite element (4DOF) has the shape

$$\{d_n\} = \{d_{n_1}, d_{n_2}, ..., d_{n_m}\}^T. \tag{13}$$

The column vectors of internal forces could be written according to relation (5) as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \frac{\partial\varphi_x}{\partial x} \\ \frac{\partial\varphi_y}{\partial y} \\ \frac{\partial\varphi_x}{\partial y}+\frac{\partial\varphi_y}{\partial x} \end{Bmatrix} = [D_M]\{\varepsilon_M\}. \tag{14}$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \frac{Ehk_Q}{2(1+\nu)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \frac{\partial w}{\partial x}+\varphi_x \\ \frac{\partial w}{\partial y}+\varphi_y \end{Bmatrix} = [D_Q]\{\varepsilon_Q\}. \tag{15}$$

Taking into account equations (12) it results

$$\{\varepsilon_M\} = [B_{M_1}\, B_{M_2} \dots B_{M_i} \dots B_{M_{mk}}] \begin{Bmatrix} d_{n_1} \\ d_{n_2} \\ \vdots \\ d_{n_m} \end{Bmatrix},$$

or

$$\{\varepsilon_M\} = \sum_{i=1}^{m} [B_{M_i}]\{d_{n_i}\}; \tag{16}$$

$$\{\varepsilon_Q\} = [B_{Q_1}\, B_{Q_2} \dots B_{Q_i} \dots B_{Q_m}] \begin{Bmatrix} d_{n_1} \\ d_{n_2} \\ \vdots \\ d_{n_m} \end{Bmatrix},$$

or

$$\{\varepsilon_Q\} = \sum_{i=1}^{m} [B_{Q_i}]\{d_{n_i}\}, \tag{17}$$

where

$$[B_{M_i}] = 0 \begin{bmatrix} 0 & \frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial y} \\ 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \text{ and } [B_{Q_i}] = 0 \begin{bmatrix} \frac{\partial N_i}{\partial x} & N_i & 0 \\ \frac{\partial N_i}{\partial y} & 0 & N_i \end{bmatrix}. \tag{18}$$

In order to solve the problem in terms of natural coordinates, relations (17) are used to obtain the derivatives of shape function, $\partial N_i/\partial \zeta$ and $\partial N_i/\partial \eta$, in terms of $\partial N_i/\partial x$ and $\partial N_i/\partial y$,

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix}, \tag{19}$$

where $[J]$ is the Jacobian matrix. Obviously

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} J^*_{11} & J^*_{12} \\ J^*_{21} & J^*_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}. \tag{20}$$

The matrices $[B_{M_i}]$ and $[B_{Q_i}]$ could be expressed in the form:

$$[B_{M_i}] = \begin{bmatrix} 0 & J^*_{11}\frac{\partial N_i}{\partial \zeta} + J^*_{12}\frac{\partial N_i}{\partial \eta} & 0 \\ 0 & 0 & J^*_{21}\frac{\partial N_i}{\partial \zeta} + J^*_{22}\frac{\partial N_i}{\partial \eta} \\ 0 & J^*_{21}\frac{\partial N_i}{\partial \zeta} + J^*_{22}\frac{\partial N_i}{\partial \eta} & J^*_{11}\frac{\partial N_i}{\partial \zeta} + J^*_{12}\frac{\partial N_i}{\partial \eta} \end{bmatrix},$$

(21)

$$[B_{Q_i}] = \begin{bmatrix} J^*_{11}\frac{\partial N_i}{\partial \zeta} + J^*_{12}\frac{\partial N_i}{\partial \eta} & N_i & 0 \\ J^*_{21}\frac{\partial N_i}{\partial \zeta} + J^*_{22}\frac{\partial N_i}{\partial \eta} & 0 & N_i \end{bmatrix}.$$

The derivatives of shape functions with respect to the natural coordinates have been computed for the two types of isoparametric finite elements:

a) Finite element with four nodes:

$$\frac{\partial N_i}{\partial \zeta} = \frac{1}{4}\zeta_i(1 + \eta_i\eta); \quad \frac{\partial N_i}{\partial \eta} = \frac{1}{4}\eta_i(1 + \zeta_i\zeta). \tag{22}$$

b) Finite element with eight nodes:

b_1) for the nodes from the corners

$$\frac{\partial N_i}{\partial \zeta} = \frac{1}{4}\zeta_i(1 + \eta_i\eta)(2\zeta\zeta_i + \eta\eta_i); \quad \frac{\partial N_i}{\partial \eta} = \frac{1}{4}\eta_i(1 + \zeta_i\zeta)(2\eta\eta_i + \zeta\zeta_i); \tag{23}$$

b_2) for the nodes from the side midpoints

$$\frac{\partial N_i}{\partial \zeta} = \frac{1}{2}\zeta_i^3(1 - \eta^2) - 2\zeta\eta_i^2(1 + \eta\eta_i); \quad \frac{\partial N_i}{\partial \eta} = \frac{1}{2}\eta_i^3(1 - \zeta^2) - \eta\zeta_i^2(1 + \zeta\zeta_i). \tag{24}$$

The global strain energy of a finite element is

$$\Pi_e = \frac{1}{2}\int_{A_e}\{\varepsilon_M\}^T[D_M]\{\varepsilon_M\}\,\mathrm{d}A + \frac{1}{2}\int_{A_e}\{\varepsilon_Q\}^T[D_Q]\{\varepsilon_Q\}\,\mathrm{d}A + U_e, \tag{25}$$

where U_e is the energy produced by exterior actions.

By using relations (14) and (15), Π_e becomes

$$\Pi_e = \frac{1}{2}\{d_n\}_e^T\left(\int_{A_e}[B_M]^T[D_M][B_M]\,\mathrm{d}A + \int_{A_e}[B_Q]^T[D_Q][B_Q]\right)\{d_n\}_e\,\mathrm{d}A + U_e. \tag{26}$$

The strain energy is minimized with respect to the nodal displacements and the final relation is

$$\{F_n\}_e = [k]_e\{d_n\}_e \tag{27}$$

where $[k]_e$ is the finite element stiffness matrix and $\{F_n\}_e$ – the finite element column vector of nodal forces.

The relation force – displacement for the whole plate is determined by following the well-known assembly procedure that takes into account the boundary conditions, without the restrictions imposed by thin plates theory.

5. Conclusions

1. The isoparametric finite elements ensure an accurate discretization of plates and by decoupling the displacements; the same functions could be used.
2. The method could be easily extended to elements with variable thickness.
3. The most important aspect of this procedure is the possibility of a unitary approach of thin and thick plates.

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ASUPRA METODEI ELEMENTULUI FINIT APLICATĂ PLĂCILOR PLANE GROASE

(Rezumat)

Se dezvoltă metoda elementului finit aplicată plăcilor plane groase acţionate de forţe transversale. Calculul ia în considerare efectul forfecării şi condiţiile la limită reale pentru rezemările clasice, incluzându-se într-un calcul unitar, atât plăcile groase cât şi cele subţiri.

Utilizarea elementelor finite izoparametrice creează facilităţi în formularea problemelor şi disponibilităţi pentru realizarea unor programe de calcul performante.