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STRESS AND STRAIN ANALYSIS IN CONTINUUM MECHANICS WITH APPLICABILITY IN SOIL MECHANICS

BY

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Abstract. The approach towards solving problems related to the construction environment is currently facilitated by the existence of advanced computers and programs developed specifically for different engineering branches. Finite element based programs are powerful tools capable of solving even the most complex engineering structures. In this context, the user dictates the quality of the results obtained from the numerical analysis. For this reason, an essential aspect for obtaining representative results for the studied problems is the user’s understanding of the mathematical basis and the stages involved in the numerical analysis process. Given that the mathematical basis is represented by the established laws of continuum mechanics, this article aims to present the process of determining the state of stress and strains of a continuous body in an easily understandable way, providing in the end an example of calculation in the field of geotechnical engineering. Understanding these basic principles is fundamental in the numerical modelling of engineering problems.

Keywords: stress; strain; tensor, numerical analysis; finite element method.

1. Introduction

With the development of advanced computer technology, considerable progress has been made in all areas of engineering and, in particular, in the civil

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engineering branches. Analytical solutions are still used for many of the problems encountered in this field. However, they are of limited applicability in case of a high level of structural complexity or if additional information is required for the problem at hand. Programs based on advanced numerical methods, such as the finite element method, eliminate many of the limitations of analytical methods. These programs are generally used by the generation of young engineers, and, because they are relatively easy to use, there is a tendency to neglect the mathematical basis and stages involved in the numerical analysis process. Understanding these basic principles is fundamental in modeling civil engineering structures and in the process of validating the results (Wood, 2004).

This article aims to guide the reader through all the steps required in order to determine the state of stress and strain at a point in a continuum body. The state of stress is detailed in the first part of the article, beginning with the definition of stress, building the stress tensor and demonstrating the symmetry of the stress tensor. The state of strain is discussed next, with the definition of normal and shear strain and building the strain tensor. Finally, a practical example of an analytical and numerical determination of the stress state is discussed for a simple problem in geotechnical engineering.

2. State of Stress

Determining the state of stress is necessary in order to assess the manner in which a body behaves under the action of forces. This will be further illustrated for an arbitrary body in equilibrium under an arbitrary system of forces (Fig. 1).

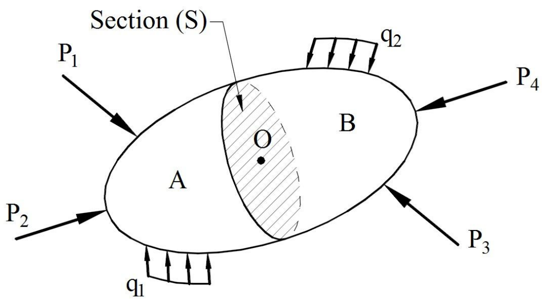

Fig. 1 – Body in equilibrium under an arbitrary system of forces (Mazilu, 1977).

2.1. Definition of Stress

Under the action of external forces, internal forces will develop between different parts of the body. In order to study these forces at any point *O*, the body will be sectioned over an area *S*, dividing it into two parts, *A* and *B*. Analysing only one of these parts (for example part *A*) it can be said that it is in equilibrium under the action of external and internal forces distributed on the surface *S*, the latter representing the action of the material of part *B* on the material of part *A*. It can be assumed that these forces are distributed over the surface *S* in the same way in which wind pressure is distributed over the surface on which it acts. The magnitude of these forces is defined by their intensity, meaning the amount of force per unit area. This intensity if called "stress" (Timoshenko & Goodier, 1951).

Stresses are not generally distributed over the surface *S* in a uniform manner. For this reason, in order to obtain the magnitude of the stress, a small area, ΔA, defined around point *O* will be extracted from the surface *S* (Fig. 2). The forces acting on the ΔA area can be reduced to a resultant noted ΔP, representing the action of the material of part *B* on part *A* for this given area. The limiting value of the ratio $\Delta P/\Delta A$ gives us the stress, *p*, acting at point *O* of the surface *S* (Timoshenko & Goodier, 1951):

$$\lim \frac{\Delta P}{\Delta A_{\Delta A \to 0}} = p. \tag{1}$$

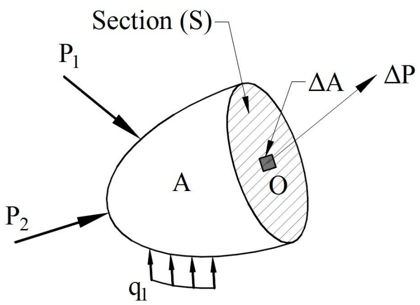

Fig. 2 – Stress development in point *O* of surface *S* (Mazilu, 1977).

Being a physical quantity defined by magnitude and direction, stress is regarded as a vector and not a scalar. As such, the total stress, *p*, can be broken down into the specific components of a vector:

a) a normal component, σ, acting along the normal direction to the surface on which $\vec{p}$ acts;

b) two tangential components, τ, perpendicular to each other, acting parallel to the surface on which $\vec{p}$ acts.

In a Cartesian coordinate system, of axes x($\vec{i}$), y($\vec{j}$) and z($\vec{k}$), the total stress $\vec{p}$ can be decomposed as presented in Fig. 3.

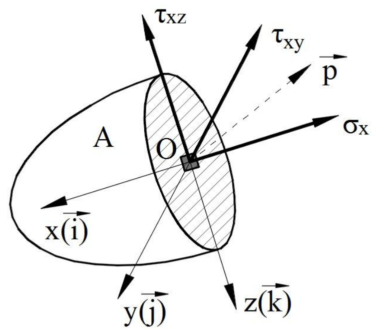

Fig. 3 – Stress components in a *x*-*y*-*z* Cartesian coordinate system (Mazilu, 1977).

The stress vector has the following expression (Mazilu, 1977):

$$\vec{p} = \sigma_x \vec{i} + \tau_{xy} \vec{j} + \tau_{xz} \vec{k}, \tag{2}$$

where: $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors of the *x*, *y* and *z* axes.

Regarding the sign convention it is generally considered that the normal stress component, *σ*, is positive when it produces tension in the body on which it acts (Timoshenko & Goodier, 1951; Mazilu, 1977). This rule however doesn't apply for all engineering branches. For example, in geotechnical engineering the normal stress components are considered positive when they produce compression (Desai & Christian, 1977). As for the tangential stresses, the following rule applies regardless of the engineering field: if the normal component, *σ*, acts in the negative direction of its corresponding axis then the tangential stresses, *τ*, will be positive if they also act in the negative direction of their corresponding axes (Timoshenko & Goodier, 1951; Mazilu, 1977). In Fig. 3 all of the stress components are represented in their positive direction.

2.2. Stress Tensor

So far, it can be concluded that through a point *O* an infinite number of planes can be chosen and each one of them has a total stress $\vec{p}$ that corresponds to it, with three components: a *σ* component and two *τ* components (Mazilu, 1977; Das, 2008).

In order to find the stress state in point *O* (which means to know the stresses that act on any plane that goes through point *O*), it is necessary and sufficient to know the stresses acting on three orthogonal planes that go through point *O* (Fig. 4) (Mazilu, 1977).

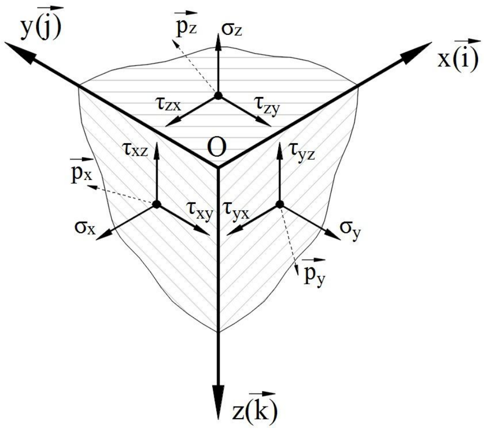

Fig. 4 – Stress components in three dimensions (Mazilu, 1977).

All nine components illustrated in Fig. 4 are positive according to the sign convention which was previously discussed and they are represented in the following form (Stanciu & Lungu, 2006; Murthy, 2007; Yu, 2006):

$$[T_\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{3}$$

$[T_\sigma]$ is called the stress tensor in point *O*, in the *x*-*y*-*z* Cartesian coordinate system. Each of the columns of the tensor represent the total stress components for the planes defined by the unit vectors $\vec{i}, \vec{j}, \vec{k}$ (Mazilu, 1977).

It can be concluded that the stress state in a random point of a body is defined by a system of vectors, leading to the notion of second-order tensor with nine components (Mazilu, 1977; Yu 2006).

One of the fundamental problems of continuum mechanics lies in determining the stress tensor for every point of the body, meaning determining the stress field for that specific body (Mazilu, 1977; Yu 2006).

2.3. Symmetry of the Stress Tensor

The symmetry of the stress tensor will be demonstrated using the principle of conservation of momentum for an elementary unit of very small dimensions *dx, dy* and *dz* (Fig. 5) that corresponds to point *O* of the body.

In order for this elementary unit to be in equilibrium, the sum of all moments of the forces acting in point *O* needs to be equal to zero:

$$\sum M(O) = 0 . \tag{4}$$

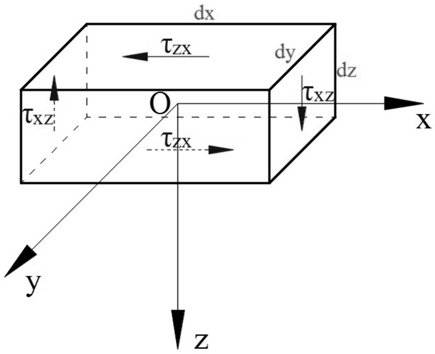

Fig. 5 – Elementary unit around point *O* (Mazilu, 1977; Das, 2008).

To illustrate this principle, the equation of equilibrium will be written for the *xz* plane. As can be observed, the normal stresses, σ, do not generate momentum and therefore will not intervene in the equation. The stress variation from one side of the elementary unit to the opposite side will also be neglected. Normally, if τ_{xz} acted on the left side, the right side would have the following stress (Mazilu, 1977):

$$\tau_{xz} + \frac{\partial \tau_{xz}}{\partial_x} dx \tag{5}$$

These variations can be neglected in the equilibrium equation as they are quantities of higher order which vanish in the limit (Mazilu, 1977).

Thus, the equilibrium equation will have the form shown in Fig. 6.

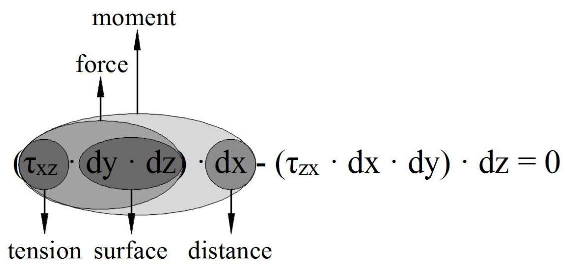

Fig. 6 – The equation of equilibrium.

The equilibrium equation requires that $\tau_{xz} = \tau_{zx}$. Similarly, it can be demonstrated that $\tau_{xy} = \tau_{yx}$ and $\tau_{yz} = \tau_{zy}$ (Yu, 2006).

Thus, the number of stress components reduces to six: three normal stresses and three tangential stresses (Yu, 2006).

3. State of Strain

An elementary unit belonging to a body will suffer displacements and deformations when that body is solicited by an unbalanced system of forces (Fig. 7).

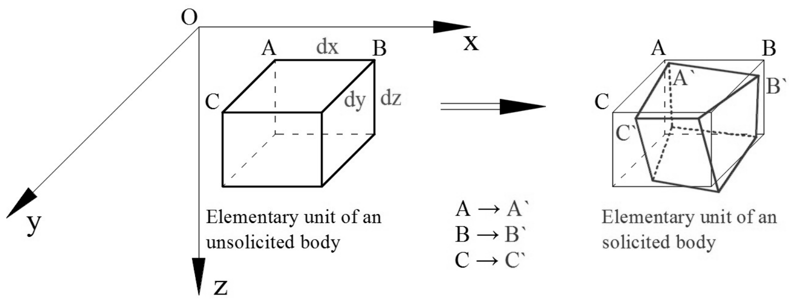

Fig. 7 – Displacement and deformation of an elementary unit of a body (Das, 2008; Budhu, 2000).

The displacement components are noted *u*, *v* and *w* and they correspond to the *x*, *y* and *z* axes. For simplification, the strain components will be illustrated only for the *xy* plane (Fig. 8).

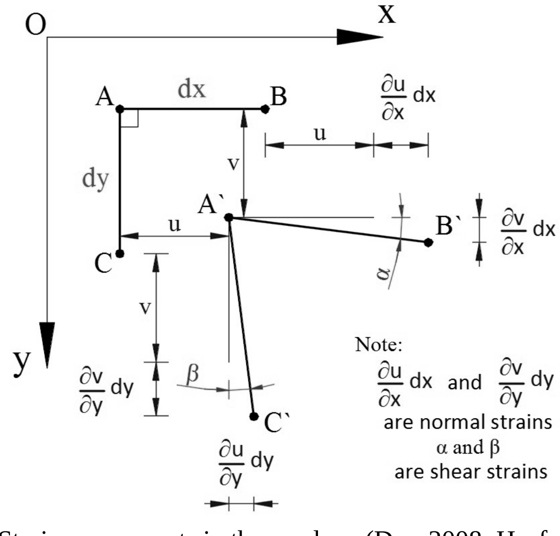

Fig. 8 – Strain components in the *xy* plane (Das, 2008; Hosford, 2010).

3.1. Normal Strain

Normal strain, ε, is the ratio given in the following equation, where l_0 is the initial length and l_1 is the final length, after deformation (Fig. 9):

$$\varepsilon = \frac{l_1 - l}{l_0} = \frac{\Delta l}{l_0} \ . \tag{6}$$

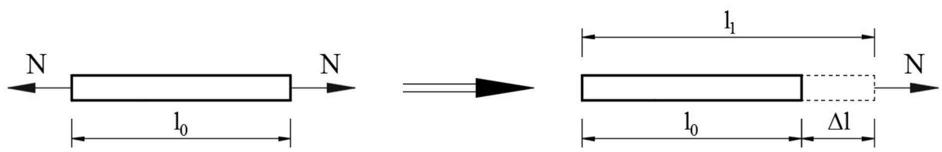

Fig. 9 – Definition of normal strain, (Hosford, 2010).

If $\varepsilon > 0 \rightarrow$ specific elongation, if $\varepsilon < 0 \rightarrow$ shortening/specific shrinkage.

For an elementary unit of dimensions dx, dy and dz (Fig. 7), the normal strains ε_x, ε_y and ε_z are determined based on the equations given below (Yu, 2006; Hosford, 2010).

$$\varepsilon_x = \frac{\frac{\partial u}{\partial x} dx}{dx} \rightarrow \varepsilon_x = \frac{\partial u}{\partial x}, \tag{7}$$

$$\varepsilon_y = \frac{\frac{\partial v}{\partial y} dy}{dy} \rightarrow \varepsilon_y = \frac{\partial v}{\partial y}, \tag{8}$$

$$\varepsilon_z = \frac{\frac{\partial w}{\partial z} dz}{dz} \rightarrow \varepsilon_z = \frac{\partial w}{\partial z}. \tag{9}$$

3.2. Shear Strain

The shear strain is represented by the angular changes that result due to deformations (Fig. 10). In the small strain hypothesis, the angle γ is approximately equal to its tangent:

$$\mathrm{tg}\gamma \approx \gamma \rightarrow \gamma = \frac{bb'}{ab}. \tag{10}$$

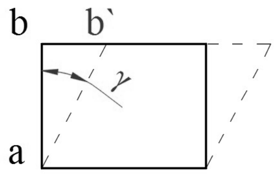

Fig. 10 – Definition of shear strain (Budhu, 2000).

The shear strain in the *xy* plane for the exemplified elementary unit from Fig. 7 is represented by the change of the angle formed between $\overline{AB}$ and $\overline{AC}$ (Fig. 8), and it is noted γ_{xy}:

$$\gamma_{xy} = \alpha + \beta \tag{11}$$

The parameters α and β are exemplified in the following equations (Das, 2008):

$$\left.\begin{aligned} \alpha \approx tg\alpha = \frac{\frac{dv}{dx}dx}{dx} \rightarrow \alpha = \frac{\partial v}{\partial x} \\ \beta \approx tg\beta = \frac{\frac{du}{dy}dy}{dy} \rightarrow \beta = \frac{\partial u}{\partial y} \end{aligned}\right\} \Rightarrow \begin{aligned} \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (12) \\ \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} \quad (13) \end{aligned}$$

Similarly, it can be demonstrated that $\gamma_{yz} = \gamma_{zy}$ and $\gamma_{xz} = \gamma_{zx}$.

The table of the strain components of an elementary unit located in the vicinity of a point belonging to a continuous body, is called strain tensor and, similar to the stress tensor, it is symmetrical with regard to the main diagonal (Stanciu & Lungu, 2006; Murthy, 2007; Yu, 2006):

$$[T_\varepsilon] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} \tag{14}$$

4. Practical Example for the Determination of the Stress Tensor

The aim is to determine the stress state and build the stress tensors that correspond to two points belonging to a loaded soil mass.

The analyzed situation is that of a continuous foundation that transmits a uniformly distributed load to the ground.

The solution is given based on an analytical and a numerical calculation. The numerical analysis was made using the finite element modeling software Plaxis 2-D.

Because the problem is analyzed from the plane point of view, the stress state is given by two normal components, σ_x and σ_z, and by two equal tangential components, $\tau_{xz} = \tau_{zx}$. Consequently, the stress tensor will be composed by a matrix with four elements.

4.1. Problem Description

The continuous foundation has a width of 2.00 m and transmits to the ground a uniform load of 100 kPa. The problem is represented schematically in Fig. 11 illustrating the position of the two points in which the state of stress will be determined.

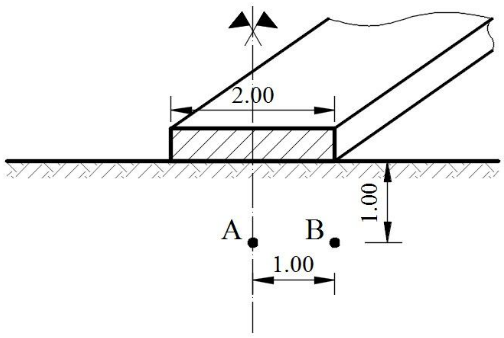

Fig. 11 – Position of the two points in which the stress state is calculated.

4.2. Analytical Solution

The stress state in a point of a soil mass is given by the stresses generated by the external forces, added to those generated by the geological pressure. All of these stresses are represented for the analyzed situation in Fig. 12.

The analytical solution for the determination of the stress state in the soil mass for a uniformly distributed continuous load, of finite width and infinite length, is given by the relations developed by Carothers (1920) for normal stresses (Eqs. 15 and 16) and for tangential stresses (Eq. 17) (Aysen, 2005; Craig, 2004; Das, 2008). The parameters involved in the calculation are shown in Fig. 13.

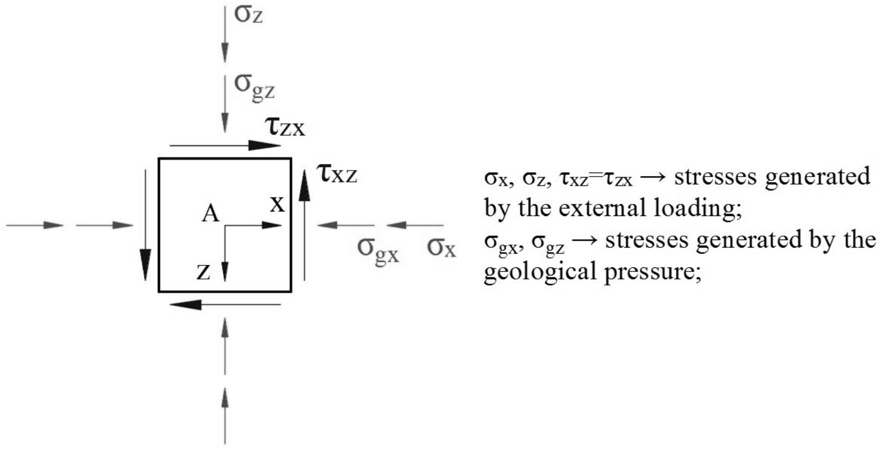

Fig. 12 – Plane state of stress in a point in the soil mass.

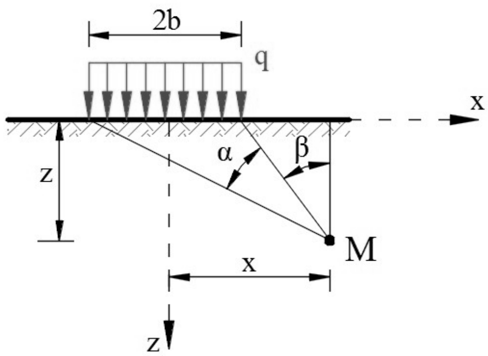

Fig. 13 – Parameters involved in the stress state calculation (Aysen, 2005; Craig, 2004; Das, 2008).

$$\sigma_x = \frac{q}{\pi} \cdot [\alpha - \sin\alpha \cdot \cos(\alpha + 2\beta)], \tag{15}$$

$$\sigma_z = \frac{q}{\pi} \cdot [\alpha + \sin\alpha \cdot \cos(\alpha + 2\beta)], \tag{16}$$

$$\tau_{xz} = \frac{q}{\pi} \cdot [\sin\alpha \cdot \cos(\alpha + 2\beta)]. \tag{17}$$

The values of the angles α and β have the following expressions (Aysen, 2005):

$$\alpha = \tan^{-1}\cdot\left(\frac{x+b}{z}\right) - \tan^{-1}\cdot\left(\frac{x-b}{z}\right), \qquad (18)$$

$$\beta = \tan^{-1}\cdot\left(\frac{x-b}{z}\right). \qquad (19)$$

The soil action generates vertical and horizontal tensions, which can be determined by the equations shown below.

$$\sigma_{gz} = \gamma \cdot z, \qquad (20)$$

$$\sigma_{gx} = \gamma \cdot z \cdot K_0, \qquad (21)$$

where: K_0 is the at rest earth pressure coefficient, which can be obtained by the equation developed by Jaky (1944), (Craig, 2004; Murthy, 2007);

$$K_0 = 1 - \sin\varphi \qquad (22)$$

where: φ is the internal friction angle of soil.

4.3. Numerical Solution

In order to validate the analytical results, a numerical model of the proposed problem was made using the finite element modeling software, Plaxis 2D.

The model was made in plane strain. Using the advantage of model symmetry, only half of the problem was introduced and analyzed into the program (Fig. 14) (Plaxis 2-D – Tutorial Manual, 2008).

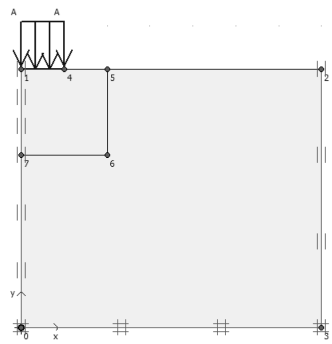

Fig. 14 – Numerical model in the finite element program, Plaxis 2-D.

The soil was modeled using the Mohr-Coulomb model with a value of Young's modulus of E = 5,000 [kN/m^2] and Poisson's ratio of ν = 0.30 [-]. The soil strength was taken into account by a value of the friction angle of φ = 15 [°] and a cohesion value of c = 30 [kN/m^2]. The volumetric weight of the soil was introduced with the value of γ = 18 [kN/m^3].

4.4. Results and Discussions

The results of the analytical and numerical calculations are presented in Fig. 15, and the stress tensors are shown in Fig. 16.

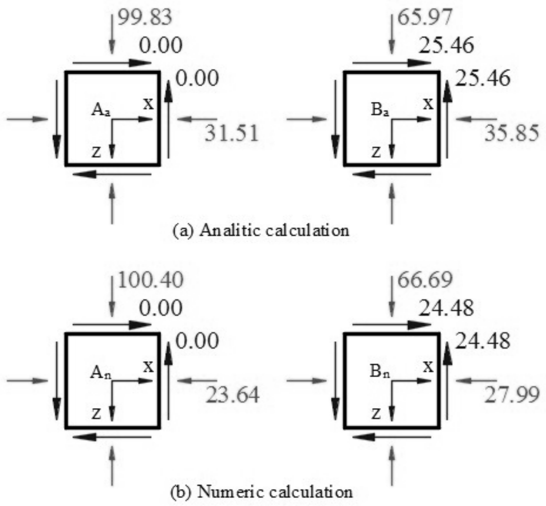

Fig. 15 – Results from analytical and numerical calculation for points A and B, [kPa].

$$[T_\sigma^{Aa}] = \begin{bmatrix} -31.51 & 0.00 \\ 0.00 & -99.83 \end{bmatrix} \qquad [T_\sigma^{Ba}] = \begin{bmatrix} -35.85 & 25.46 \\ 25.46 & -65.97 \end{bmatrix}$$

$$[T_\sigma^{Aa}] = \begin{bmatrix} -23.64 & 0.00 \\ 0.00 & -100.40 \end{bmatrix} \qquad [T_\sigma^{Ba}] = \begin{bmatrix} -27.99 & 24.48 \\ 24.48 & -66.69 \end{bmatrix}$$

Fig. 16 – Stress tensors based on the determined values, [kPa].

The negligible differences observed between the analytical and numerical results in the case of the vertical stress (σ_z) and tangential stresses ($\tau_{xz} = \tau_{zx}$) can be attributed to the level of discretization of the numerical model (Teodoru & Muşat, 2009) and to the fact that the numerical analysis gives an approximate solution and not an exact one. The differences obtained for the horizontal stress (σ_x) are due to the fact the analytical relations given by the eqs.

(15), (16) and (17) are independent of the material parameters. In the numerical calculation however, the horizontal stress (σ_x) is influenced by the value of Poisson's ratio (ν).

5. Conclusions

This article has presented the theoretical basis behind the determination of the state of stress and strain in a point belonging to a continuous medium. The theory lies at the foundation of numerous numerical modelling programs used in various domains of construction engineering. Thus, understanding it is fundamental in the modelling process and for validation of the obtained results.

In the analysed practical example, the stress state in two pints (A and B) of a loaded soil mass has been determined, based on analytical and numerical calculations.

It has been observed that the differences between the analytical and numerical solutions are negligible for the vertical and tangential stresses. The differences found for the horizontal stress (σ_x) are attributed to the value of Poisson's ratio (ν) which in the case of the considered analytical equations is not taken into account. At the same time, the stresses are independent of the value of Young's modulus (E).

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* * * Plaxis 2D – Version 9.0, *Tutorial Manual*, Ed. Brinkgreve R.B.J., Broere W., Waterman D., The Netherlands, 2008.

ANALIZA STĂRII DE TENSIUNI ŞI DEFORMAȚII ÎN MECANICA MEDIULUI CONTINUU CU APLICABILITATE ÎN MECANICA PĂMÂNTURILOR

(Rezumat)

Abordarea problemelor din mediul construcțiilor este facilitată în prezent de existența calculatoarelor şi a programelor specializate pe diferite ramuri ale ingineriei. Programele bazate pe metoda elementului finit reprezintă instrumente puternice capabile să rezolve până şi cele mai complexe structuri inginereşti. În acest context utilizatorul este cel care dictează calitatea rezultatelor obținute în urma analizelor numerice. Din acest motiv, un aspect esențial pentru obținerea unor rezultate reprezentative pentru problemele studiate îl reprezintă înțelegerea de către utilizator a bazei matematice şi a etapelor de rezolvare parcurse în cadrul programelor de calcul. Având în vedere că această bază matematică este reprezentată de legile consacrate ale mecanicii mediului continuu, articolul îşi propune să prezinte într-un mod uşor de înțeles procesul de determinare a stării de tensiuni şi deformații a unui corp continuu, oferind în final un exemplu de calcul. Înțelegerea acestor principii de bază este fundamentală în procesul de modelare numerică a problemelor inginereşti.