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A FINITE ELEMENT STUDY OF THE BENDING BEHAVIOR OF BEAMS RESTING ON TWO-PARAMETER ELASTIC FOUNDATION

BY

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Although the Winkler's model is a poor representation of the many practical subgrade or subbase materials, it is widely used in soil-structure problems for almost one and a half century. The foundations represented by Winkler model can not sustain shear stresses, and hence discontinuity of adjacent spring displacements can occur. This is the prime shortcoming of this foundation model which in practical applications may result in significant inaccuracies in the evaluated structural response. In order to overcome these problem many researchers have been proposed various mechanical foundation models considering interaction with the surroundings. Among them we shall mention the class of two-parameter foundations - named like this because they have the second parameter which introduces interactions between adjacent springs, in addition to the first parameter from the ordinary Winkler's model. This class of models includes Filonenko-Borodich, Pasternak, generalized, and Vlasov foundations. Mathematically, the equations to describe the reaction of the two-parameter foundations arc equilibrium ones, and the only difference is the definition of the parameters. For the convenience of discussion, the Pasternak foundation is adopted in present paper.

In order to analyse the bending behavior of a Euler-Bernoulli beam resting on two-parameter elastic foundation a (displacement) Finite Element (FE) formulation, based on the cubic displacement function of the governing differential equation, is introduced. The resulting effects of shear stiffness of the Pasternak model on the mechanical quantities are discussed in comparison with those of the Winkler's model. Some numerical case studies illustrate the accuracy of the formulation and the importance of the soil shearing effect in the vertical direction, associated with continuous elastic foundation.

1. Introduction

The concept of beams and slabs resting on elastic foundations has been extensively used by geotechnical, pavement and railroad engineers for foundation design and analysis. The analysis of structures resting on elastic foundations is usually based on a relatively simple model of the foundation's response to applied loads.

Generally, the analysis of bending of beams resting on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional, at every point, to the deflection of the beam at that point. The vertical deformation characteristics of the foundation are defined by means of continuous, closely spaced linear springs. The constant of proportionality of these springs is known as the modulus of subgrade reaction, k_0. This simple representation of elastic foundation was introduced by Winkler in 1867. The Winkler's model (one parameter model), which has been originally developed for the analysis of railroad tracks, is very simple but does not accurately represent the characteristics of many practical foundations. One of the most important deficiencies of the Winkler's model is that a displacement discontinuity appears between the loaded and the unloaded part of the foundation surface. In reality, the soil surface does not show any discontinuity (Fig. 1).

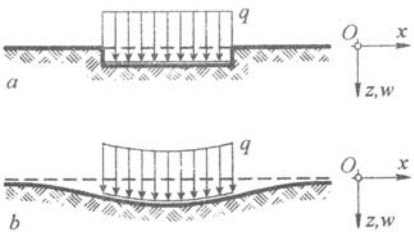

Fig. 1.- Deflections of elastic foundations under uniform pressure: *a* – Winkler's foundation; *b* – practical soil foundation.

In order to eliminate the deficiency of Winkler's model, improved theories have been introduced on refinement of Winkler's model, by visualizing various types of interconnections such as shear layers and beams along the Winkler springs [5] (Filonenko-Borodich (1940), Hetenyi (1946), Pasternak (1954), Vlasov and Leontiev (1960), Kerr (1964)). These theories have been attempted to find an applicable and simple model of representation of foundation medium. The two-parameter Pasternak's model [5] is one of them.

Two-parameter foundation models are more accurate than the one-parameter (*e.g.* Winkler) foundation model. As a special case if the second parameter is neglected, the mechanical modeling of the foundation using the Pasternak's formulation converges to the Winkler's formulation.

The two-parameter Pasternak foundation assumes the existence of shear interaction between the spring elements. This may be accomplished by connecting the ends of the springs with a beam consisting of incompressible vertical elements which deform only by transverse shear (Fig. 2). The stiffness of the springs and the shear rigidity of this beam are the two parameters of the foundation.

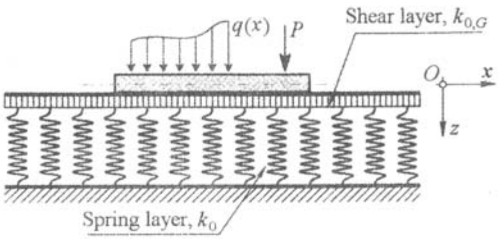

Fig. 2.- Pasternak's elastic foundation.

Of all available elastic foundation models, the Pasternak's one is the most natural extension of the Winkler's model for homogeneous foundation soil, when the second parameter, shear modulus, G, is considered in the analysis [5].

2. Basic Assumptions and Analytical Formulation

In what follows we consider straight beams with constant section loaded by forces placed in a principal plane of inertia and continuously supported on a deformable elastic foundation. Beam material is linearly elastic, homogeneous, isotropic and continuous. The foundation medium is assumed to be linear, homogeneous and isotropic.

The considered beam, supported by a Pasternak foundation having spring and shear stiffnesses, k_0 and $k_{0,G}$, respectively, is represented in Fig. 3. The reactive pressure of the two-parameter foundation subjected to a distributed load, $q(x)$, is described by [5]

$$(1)\qquad p(x) = k_0 B w(x) - k_{0,G} B \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = k w(x) - k_G \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2},$$

where: B is the width of the beam cross section; w – deflection of the centroidal line of the beam.

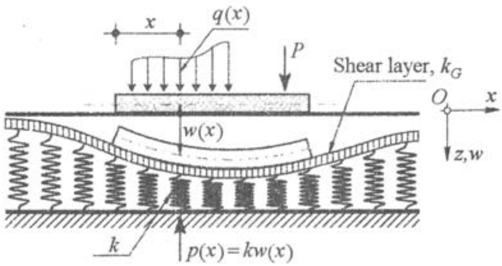

Fig. 3.– Beam resting on Pasternak's elastic foundation

The governing equations of the centroidal line of the deformed beam resting on elastic foundation is [7]

$$(2)\qquad EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x) - p(x),$$

or substituting $p(x)$ from (1)

$$(3)\qquad EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + k w(x) - k_G \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = q(x),$$

where E is the modulus of elasticity of the constitutive material of the beam; I – the moment of inertia for the cross section of the beam.

3. Finite Elements Formulation

The assumptions and restrictions underlying the development are the same as those of elementary beam theory with the addition of the following hipothesis:

1. the element is of length l and has two nodes, one at each end;
2. the element is connected to other elements only at the nodes;
3. element loading occurs only at the nodes.

The beam is divided into m unidimensional finite elements (FE) and to each i node of their interconnection, two degrees of freedom are allowed: D_{iw} – the vertical displacement and $D_{i\theta}$ – the slope of cross section. The $\{D\}$ vector of positive nodal displacements is build just like in the system of xOz general axes from Fig. 4. In the same way the vector of external nodal actions is build namely

$$(4)\quad \{D\} = \{D_{1w}\, D_{1\theta} ... D_{iw}\, D_{i\theta} ... D_{nw} D_{n\theta}\}^T,\ \{P\} = \{P_{1w}\, P_{1\theta} ... P_{iw}\, P_{i\theta} ... P_{nw} P_{n\theta}\}^T.$$

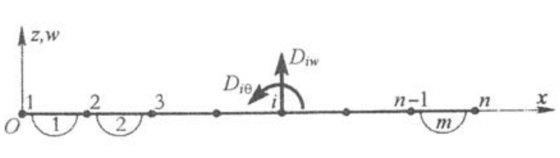

Fig. 4.– FE discretization of the beam domain.

To each one dimensional element of beam type, two degrees of fredom are allowed at both extremities: deflection, w_1 and slope, θ_1 and w_2, θ_2, respectively, positives in the system of local axes from Fig. 5.

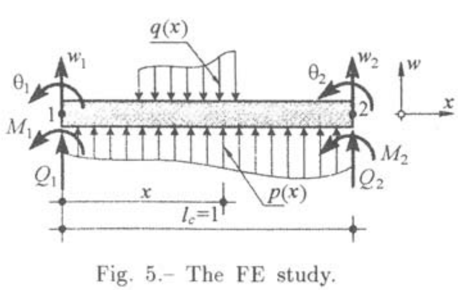

Fig. 5.– The FE study.

With help of these displacements, the $\{d_e\}$ vector of elemental nodal displacements and, similarly, the $\{S_e\}$ vector of elemental nodal forces, with respect to the system of local axes, are defined

$$(5)\qquad \{d_e\} = \{w_1\ \theta_1\ w_2\ \theta_2\}^T,\ \{S_e\} = \{Q_1\ M_1\ Q_2\ M_2\}^T.$$

We must note that Q_1 and Q_2 from (5) are not simply the transverse shear forces in the beam; they includes also the shear resistance associated with modulus, k_G, of the two-parameter foundation. Forces Q_i, (i= 1, 2), are generalized shear forces defined by

$$Q_i = V_i + V_i^*, \tag{6}$$

where $V_i = EI\,\mathrm{d}^3w(x)/\,\mathrm{d}x^3$ is the usual shear contribution from elementary beam theory; $V_i^* = -k_G\,\mathrm{d}w(x)/\,\mathrm{d}x$ – the shear contribution from Pasternak's foundation (negative sign arises because a positive slope requires opposite shear forces in the foundation).

Considering the four boundary conditions and the one-dimensional nature of the problem in terms of the independent variable, we assume the displacement function in the form

$$w_e(x) = a_0 + a_1x + a_2x^2 + a_3x^3. \tag{7}$$

The choice of a cubic function to describe the displacement is not arbitrary. With the specification of four boundary conditions, we can determine no more than four constants in the assumed displacement function. The second derivative of the assumed displacement function, $w_e(x)$, is linear; hence, the bending moment varies linearly, at most, along the length of the element. This is in accord with the assumption that loads are applied only at the element nodes.

Applying the boundary conditions

$$\begin{cases} w_e(x=x_1) = w_1, \;\; w_e(x=x_2) = w_2, \\ \left.\dfrac{\mathrm{d}w_e}{\mathrm{d}x}\right|_{x=x_1} = \theta_1, \;\; \left.\dfrac{\mathrm{d}w_e}{\mathrm{d}x}\right|_{x=x_2} = \theta_2, \end{cases} \tag{8}$$

successively, yields

$$\begin{cases} w_e(x=0) = w_1 = a_0, \\ w_e(x=l) = w_2 = a_0 + a_1l + a_2l^2 + a_3l^3, \\ \left.\dfrac{\mathrm{d}w_e}{\mathrm{d}x}\right|_{x=0} = \theta_1 = a_1, \\ \left.\dfrac{\mathrm{d}w_e}{\mathrm{d}x}\right|_{x=l} = \theta_2 = a_1l + 2a_2l + 3a_3l^2. \end{cases} \tag{9}$$

Solving the equations system (9) the coefficients of displacement function in terms of the nodal variables are obtained, which are substitute in (7) to determine the expression of the deflection *i.e.*

$$w_e(x) = N_1(x)w_1 + N_2(x)\theta_1 + N_3(x)w_2 + N_4(x)\theta_2 = [N_i]^T\{d_e\}, \tag{10}$$

or in reduced variable ($\xi = x/l$)

$$w_e(\xi) = N_1(\xi)w_1 + N_2(\xi)\theta_1 + N_3(\xi)w_2 + N_4(\xi)\theta_2 = [N_i]^T\{d_e\}, \tag{11}$$

where $N_i(x), (i = 1, ..., 4)$, are the interpolation functions (of Hermite type) that describe the distribution of displacement in terms of nodal values in the nodal displacement vector $\{d_e\}$

$$\tag{12} \begin{cases} N_1(x) = N_1(\xi) = 1 - 3\dfrac{x^2}{l^2} + 2\dfrac{x^3}{l^3} = 1 - 3\xi^2 + 2\xi^3, \\ N_2(x) = N_2(\xi) = x - 2\dfrac{x^2}{l} + \dfrac{x^3}{l^2} = l(\xi - 2\xi^2 + \xi^3), \\ N_3(x) = N_3(\xi) = 3\dfrac{x^2}{l^2} - 2\dfrac{x^3}{l^3} = 3\xi^2 - 2\xi^3, \\ N_4(x) = N_4(\xi) = -\dfrac{x^2}{l} + \dfrac{x^3}{l^2} = -l(\xi^2 - \xi^3). \end{cases}$$

As the polynomial (10) represents an approximate solution of the governing equations (3), it result the residuum (error or discrepancy)

$$\tag{13} \varepsilon(x) = EI\frac{\mathrm{d}^4 w_e(x)}{\mathrm{d}x^4} - k_G\frac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2} + kw_e(x) - q(x) \neq 0.$$

The minimizing of this residuum means to the annulment of Galerkin balanced functional, where the weight is considered for each of the four functions, $N_i(x)$, $(i=1, 2,...,4)$

$$\Pi_e = \int_0^l N_i(x)\varepsilon(x)\,\mathrm{d}x =$$

$$\tag{14} \begin{aligned} &= \int_0^l N_i(x)\left[EI\frac{\mathrm{d}^4 w_e(x)}{\mathrm{d}x^4} - k_G\frac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2} + kw_e(x) - q(x)\right]\mathrm{d}x = \\ &= EI\int_0^l N_i(x)\frac{\mathrm{d}^4 w_e(x)}{\mathrm{d}x^4}\,\mathrm{d}x - k_G\int_0^l N_i(x)\frac{\mathrm{d}^2 w_e(x)}{\mathrm{d}x^2}\,\mathrm{d}x + \int_0^l N_i(x)kw_e(x)\,\mathrm{d}x - \\ &\quad - \int_0^l N_i(x)q(x)\,\mathrm{d}x = 0, \ \ (i = 1, ..., 4). \end{aligned}$$

In first integral from (14), utilizing the parts procedure twice and taking into account the differential relations (in FEM sign convention) from elementary beam theory

$$\tag{15} \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M(x)}{EI}, \ \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = \frac{Q(x)}{EI},$$

we obtain

$$(16)\quad \begin{aligned}\Pi_e = N_i(x)Q(x)\Big|_0^l + N_i(x)M(x)\Big|_0^l + EI\int_0^l N_i''(x)w_e''(x)\,\mathrm{d}x-\\ -k_G\int_0^l N_i(x)w_e''(x)\,\mathrm{d}x + k\int_0^l N_i(x)w_e(x)\,\mathrm{d}x - \int_0^l N_i(x)q(x)\,\mathrm{d}x = 0,\ (i=1,...,4).\end{aligned}$$

In second integral from (16), utilizing the parts procedure once and taking into account the relations (6) we obtain

$$(17)\quad \begin{aligned}\Pi_e = N_i(x)V(x)\Big|_0^l + N_i(x)M(x)\Big|_0^l + EI\int_0^l N_i''(x)w_e''(x)\,\mathrm{d}x+\\ +k_G\int_0^l N_i'(x)w_e'(x)\,\mathrm{d}x + k\int_0^l N_i(x)w_e(x)\,\mathrm{d}x - \int_0^l N_i(x)q(x)\,\mathrm{d}x = 0,\ (i=1,...,4),\end{aligned}$$

or in matrix notation,

$$(18)\quad ([k_e]+[k_{e,k}]+[k_{e,G}])\{d_e\} = \{S_e\} - \{R_e\},$$

were: $[k_e]$ is the stiffness matrix of the flexure beam element; $[k_{e,k}]$ – the stiffness matrix of springs layer; $[k_{e,G}]$ – the stiffness matrix of shear layer; $\{R_e\}$ – the reactions vector of double embeded beam from distributed loads on the element.

The last relation represents the elemental physical relation of the one-dimensional finite element of beam resting on Pasternak elastic foundations.

The terms of $[k_e]$ matrix are calculated using the relation [2],...,[4]

$$(19)\quad [k_e] = EI\int_0^l [N_i''(x)\{N_1''\, N_2''\, N_3''\, N_4''\}]\,\mathrm{d}x = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}.$$

The terms of $[k_{e,k}]$ matrix are calculated using the generic relation [4]

$$(20)\quad [k_{e,k}] = k\int_0^l N_i(x)N_j(x)\,\mathrm{d}x = kl\int_0^l N_i(\xi)N_j(\xi)\,\mathrm{d}\xi\ \ (i,j=1,...,4),$$

resulting

$$(21)\qquad [k_{e,k}] = \frac{kl}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}.$$

The terms of $[k_{e,G}]$ matrix are calculated using the generic relation

$$(22)\qquad [k_{e,G}] = k_G \int_0^l N_i'(x)N_j'(x)\,\mathrm{d}x = k_G l \int_0^l N_i'(\xi)N_j'(\xi)\,\mathrm{d}\xi \ \ (i,j = 1,...,4),$$

obtaining [8]

$$(23)\qquad [k_{e,G}] = \frac{k_G}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}.$$

The vector $\{R_e\}$ depends on the distributed load on the element and, for $q(x) = $ $= q = \text{const.}$, it result

$$(24)\qquad \{R_e\} = \int_0^l N_i(x)q(x)\,\mathrm{d}x = \left\{\frac{ql}{2}\ \ \frac{ql^2}{12}\ \ -\frac{ql}{2}\ \ -\frac{ql}{12}\right\}^T.$$

4. Computational Example and Comparative Analysis

The effect of the shear stiffness of the foundation, as reflected by the Pasternak's two-parametric model, as well as the properties of the beam element supported by a subgrade, will be demonstrated by an example of a foundation beam in Fig. 6. Numerical calculations using Finite Element Method (FEM) has been performed using 50 elements.

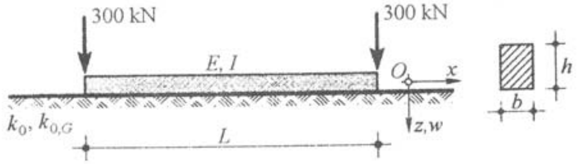

Fig. 6.- Loading condition for the beam resting on elastic foundation.

Five different sets of the material characteristics of the foundation, given in Table 1, will be studied. For all studies cases the Young's modulus of the beam material is $E = 24,000$ MPa.

Table 1

Sets of the Foundation Properties

	A	B	C	D	E
k_0, [MN/m^3]	300	500	700	1,000	1,200
$k_{0,G}$, [MN/m]	$0.35\ \mathrm{m}^2 \cdot k_0$ [1]				

Each beam cross section recorded in Table 2 was analysed for each foundation properties from Table 1 and for each length from Table 3.

Table 2

Sets of Beams Cross Sections

	I1	I2	I3	I4
b, [cm]	40	40	40	40
h, [cm]	40	60	80	100

Table 3

Sets of Beam Length

	L1	L2	L3	L4
L, [cm]	200	300	400	500

The obtained results are summarized in following figures: Figs. 7,...,10 shows the relative bending moment (percent variation of the bending behavior for the beam on Pasternak's foundation with respect to the same beam on Winkler's foundation) for each beam length recorded in Table 3. Maximum relative bending moment with respect to subgrade reaction values, is ploted in Figs. 11,...,14 for $L = 200, 300, 400$ and 500 cm, respectively.

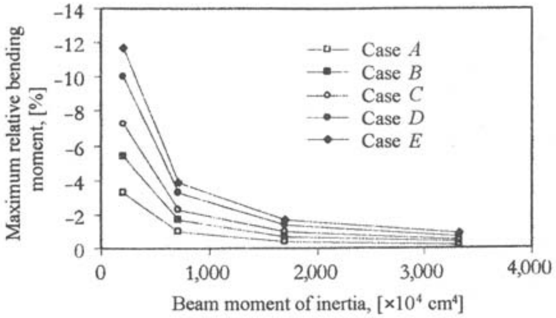

Fig. 7.- Maximum relative bending moment with respect to beam moment of inertia for L = 200 cm.

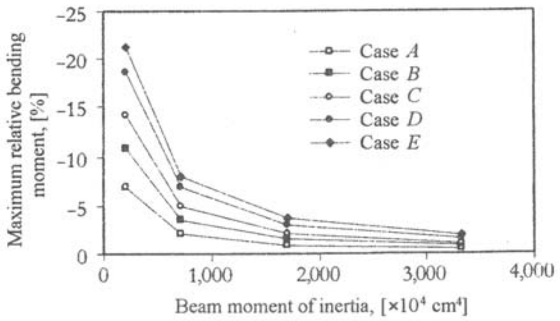

Fig. 8.– Maximum relative bending moment with respect to beam moment of inertia for L = 300 cm.

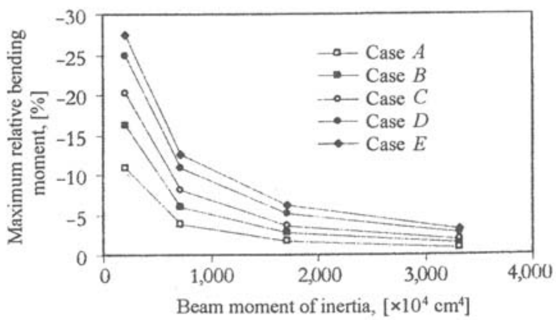

Fig. 9.– Maximum relative bending moment with respect to beam moment of inertia for L = 400 cm.

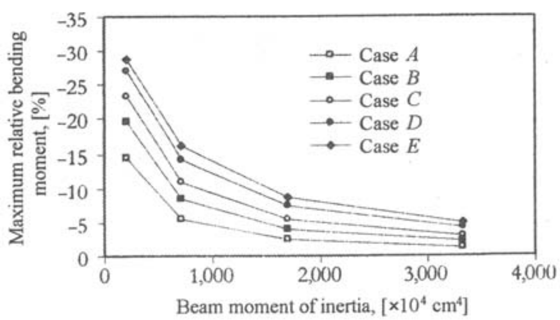

Fig. 10.– Maximum relative bending moment with respect to beam moment of inertia for L = 500 cm.

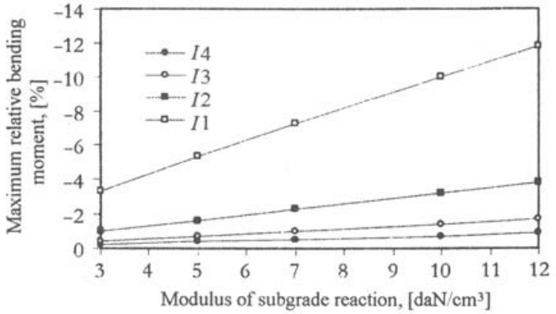

Fig. 11.– Maximum relative bending moment with respect to subgrade reaction values, for L = 200 cm.

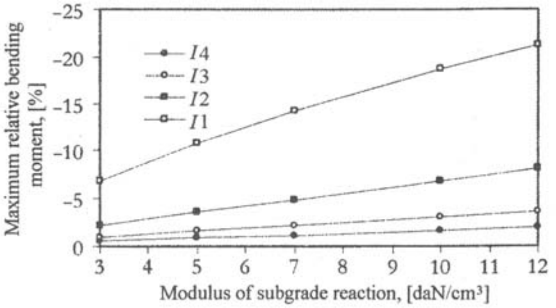

Fig. 12.– Maximum relative bending moment with respect to subgrade reaction values, for L = 300 cm.

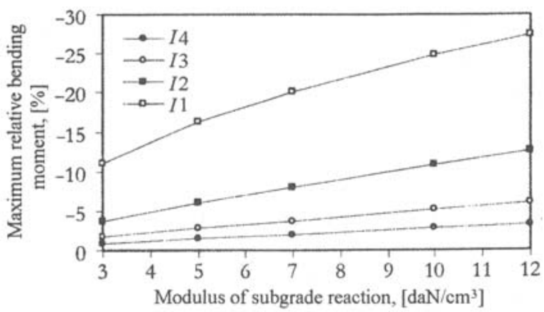

Fig. 13.– Maximum relative bending moment with respect to subgrade reaction values, for L = 400 cm.

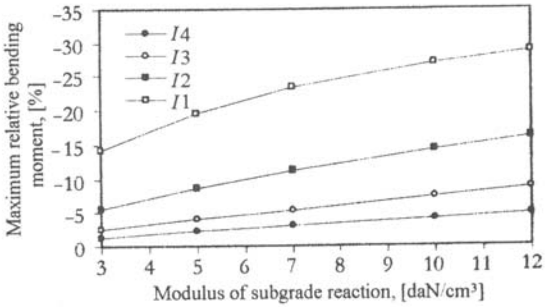

Fig. 14.– Maximum relative bending moment with respect to subgrade reaction values, for $L = 500$ cm.

5. Conclusions and Comments

Numerical calculations are carried out in order to obtain the bending behavior of the beams resting on two-parameter foundation and to clarify the effect of Pasternak's foundation.

The effects of the second parameters of the foundation is analysed and given in figures. A formulation of the two-node beam finite element on the two-parameter elastic foundation model, based on the cubic shape function of a regular flexure beam element and adding the contribution of the foundation as element foundation stiffness matrices, is also presented.

The Pasternak's foundation is a more realistic representation of foundation medium than Winkler's model but the difficulty in use of model consist in relating the model coefficients to soil parameters; experimental values for the second foundation parameter, k_G, are not provided in the literature and thus, the only available method for an analytical determination of the second parameter was found in [1].

The main conclusions of the paper can be summarized as follows:

1. When the length-to-height ratio of the beam foundation is relatively small ($L/h < 4$) the beam can be analysed as if it rests on Winkler's foundation; relative difference in bending moment due the shear effect of the Pasternak's foundation is smaller than 5% and thus can be neglected.

2. Error caused by ignoring k_G may be appreciable (30% in some cases) for large length-to-height ratio.

3. The effect of the foundation shear stiffness is more accentuate for relative large stiffness (> 300 MN/m^3) hence the Pasternak model is more adequate for rocky or gravelly soils.

4. For the practical foundation beams, cross sections are sets by recomandations prescribed in the Design Codes; in major cases length-to-height ratio has values

comprised between 3 and 6, hence soil shearing effect will affect the bending behavior with no more than 10%.

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UN STUDIU PRIN ELEMENT FINIT AL COMPORTĂRII LA ÎNCOVOIERE A GRINZILOR REZEMATE PE MEDIU ELASTIC CU DOI COEFICIENŢI DE RIGIDITATE

(Rezumat)

Deşi constituie o reprezentare nesatisfăcătoare pentru multe medii de fundare, modelul Winkler este cu precădere utilizat, de aproape un secol şi jumătate, în problemele de interacţiune teren – structură. Mediile de fundare, reprezentate prin modelul Winkler, nu pot prelua eforturi de forfecare, producându-se astfel o discontinuitate în deplasările resorturilor învecinate. Acesta constituie principalul dezavantaj al modelului Winkler, dezavantaj care în practică poate duce la erori semnificative în ceea ce priveşte răspunsul structural. Pentru a înlătura neajunsurile acestui model, numeroşi cercetători au propus diverse modele pentru terenul de fundare, modele care ţin cont de interacţiunea cu terenul învecinat fundaţiei. Dintre acestea trebuie, menţionate cele cuprinse în grupa modelelor de teren cu doi coeficienţi de rigiditate – numite astfel deoarece au un al doilea coeficient de rigiditate (ce surprinde interacţiunea dintre resorturi), suplimentar celui din clasicul model Winkler. Din această grupă de modele ale terenului de fundare fac parte: modelul Filonenko-Borodich, modelul Pasternak, modelul generalizat şi modelul Vlasov. Din punct de vedere matematic ecuaţiile care descriau aceste modele sunt ecuaţii de echilibru şi singura diferenţă constă în definirea parametrilor caracteristici. Pentru asigurarea simplităţii în formulare, în lucrare este tratat modelul Pasternak.

Pentru a analiza comportarea la încovoiere a unei grinzi Euler-Bernoulli (deformaţii numai din încovoiere), sprijinită pe mediu elastic modelat cu doi parametri, se introduce o formulare prin Element Finit bazată pe aproximarea câmpului de deplasări printr-un polinom de gradul trei. Efectul rigidităţii la forfecare al modelului Pasternak este pus în evidenţă prin comparare cu modelul Winkler. Studiile de caz reliefează acurateţea formulării şi importanţa efectului de forfecare în direcţie verticală asociat cu mediul elastic de fundare.